

# Natural Deduction Calculus for Quantified Propositional Linear-time Temporal Logic (QPTL)

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## Abstract

We present a natural deduction calculus for the quantified propositional linear-time temporal logic (QPTL) and prove its correctness. The system extends previous natural deduction constructions for the propositional linear-time temporal logic. These developments open the prospect to adapt search procedures developed for the earlier natural deduction systems and to apply the new system as an automatic reasoning tool in a variety of applications capturing more sophisticated specifications due to the expressiveness of QPTL.

## 1 Introduction

In this paper we continue our investigation of natural deduction framework for non-classical setting, this time tackling propositional linear-time temporal logic extended with propositional quantification [Sistla (1983)]. We follow the notation adopted in [French and Reynolds (2002)] calling this logic QPTL. Our construction naturally extends previously defined system for PLTL [Bolotov et al. (2006)] by new rules to capture the propositional quantification in the setting of linear time.

While the propositional quantification does not add any expressiveness to the classical logic QPTL is more expressive than PLTL presenting the same potential of expressiveness as linear-time  $\mu$ -calculus (linear-time propositional temporal fixpoint logic) [Kaivola (1997)], ETL (propositional linear-time temporal logic extended with automata constraints) [Wolper (1981)] and SIS (second order logic of one successor) [Kaivola (1997)] so that each of these formalisms is as expressive as Buchi Automata [Büchi (1962)]. A well-known example distinguishing the expressiveness of these formalisms comparing to PLTL is their ability to "count", for example, to express that some property occurs at every even moment of time [Wolper (1981)]. Nevertheless, each of these logics uses its own specific syntax and it makes sense to consider how easy these logics can be used in specification. We believe that in this list QPTL indeed occupies a special place. For example, ETL and linear time  $\mu$ -calculus formulae are very difficult for understanding. It is known, in particular, that formulae with nested fixpoints very quickly become incomprehensible while automata constraints added to the logic are far too complex to apprehend intuitively.

The language of QPTL uses the a set, *Prop*, of atomic propositions:  $p, q, r, \dots, p_1, q_1, r_1, \dots, p_n, q_n, r_n, \dots$ ; Boolean operations; temporal operators:  $\square$  – ‘always in the future’;  $\diamond$  – ‘at sometime in the future’;  $\bigcirc$  – ‘at the next moment in time’; and propositional quantifiers  $\forall$  (‘for all’) and  $\exists$  (‘there exists’).

The set of *well-formed formulae* of QPTL,  $wff_{QPTL}$  is defined as follows.

1. If  $A$  is  $wff_{PLTL}$ , then  $A$  is  $wff_{QPTL}$ .
2. If  $A$  is in  $wff_{QPTL}$  and  $\alpha$  is in Prop then  $\exists \alpha A$  are in  $wff_{QPTL}$ .

Note that  $\forall x A = \neg \exists \neg A$  and that the  $\mathcal{U}$  operation is expressible in QPTL Kaivola (1997). For the semantics of QPTL we utilise the notation of Fisher et al. (2001): it is discrete, linear sequence of states

$$\sigma = s_0, s_1, s_2, \dots$$

which is isomorphic to the natural numbers,  $\mathcal{N}$ , and where each state,  $s_i$ ,  $0 \leq i$ , consists of the propositions that are true in it at the  $i$ -th moment of time. Let  $\sigma'$  be an  $x$ -variant of  $\sigma$ . For the propositional quantifiers we evaluate a formula  $A$  in  $\sigma$  at the moment  $i \in \mathcal{N}$  as follows  $\langle \sigma, i \rangle \models \exists x A \Leftrightarrow \langle \sigma', i \rangle \models A$ , for some  $\sigma'$ . A well-formed formula,  $A$ , is satisfiable if, and only if, it is satisfiable in some model, and is valid if satisfied in every possible model,

As QPTL combines the propositional linear time logic with propositional quantification, we work, on the one hand, in our old framework of natural deduction calculi originally developed by Gentzen [Gentzen (1969)] and Jaskowski

[Jaskowski (1967)] and improved by Fitch [Fitch (1952)] and Quine [Quine (1950)]. On the other hand, we build on our previous ND constructions for the logic PLTL [Bolotov et al. (2006)] and first order logic [Bolotov et al. (2005)]. Namely, the rules for the linear-time framework are adopted from the former paper while the ideology for the rules for the propositional quantifiers are taken from the latter. The new rules, for the propositional quantifiers, allow us to decompose formulae eliminating either the  $\forall$  (freely) or  $\exists$  (with some obvious restrictions) propositional quantifier from the formula or, on the contrary, to synthesise formulae introducing these quantifiers:  $\forall$  (with some restrictions) or  $\exists$  (without restrictions). Simple example in the setting of classical logic would be to derive a valid formula  $\exists x.p \Rightarrow (x \Rightarrow p)$  from  $p \Rightarrow (q \Rightarrow p)$ . Another, perhaps not obvious, example of a valid formula is  $\exists x.x$ . Note that in the absence of the temporal operations formula  $\exists x.A = A(x/\top) \vee A(x/\perp)$ , i.e. in the classical setting propositional quantification is expressible in classical logic. An interesting example of a valid formula with temporal operators would be  $\exists x.x \wedge \bigcirc \square \neg x$  which can be interpreted as ‘now will never happen again’ [French (2003)]. Obviously, this formula is valid only in the semantics without repeating states.

To capture these and other specific QPTL properties, such as quantified induction, in our construction we introduce a range of new rules, such as for example, a 0-premise rule  $\vdash \exists x.x$ . We show that our ND rules are sound and that every theorem of the axiomatics for QPTL from French and Reynolds (2002) is a theorem in our system thus providing a completeness argument.

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