Context Dependence in Multiagent Resource Allocation*

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Abstract

A standard assumption in studies of multiagent resource allocation problems is that the value an individual agent places on its assignment remains unchanged by any redistribution of the remaining resources among the other agents. This assumption renders impossible analyses of scenarios where the utility an agent attaches to a particular set of resources is determined by factors other than the resource set itself. Thus an agent's perception of what its allocation is worth may be tempered by its view of what other agents in the system may own, e.g. if working within a coalition a particular allocation may assume a greater value if other coalition members hold certain resources. In this paper we develop a model for examining such *context dependent* valuations and consider various decision problems related to the existence of context dependent allocations satisfying various criteria.

1 Introduction

Mechanisms for reasoning about allocations of resources within a group of agents form an important body of work within the study of multiagent systems. Typical abstract models derive from game-theoretic perspectives in economics and among the issues that have been addressed are strategies that agents may use to negotiate, e.g. [9, 11, 12], and protocols for negotiation in agent societies, e.g. [3, 5, 6, 10]. A formal definition of the standard resource allocation setting is given in Section 2 below, however, the analyses of this paper arise from one particular aspect of this model. An implicit assumption it makes is that the value an agent, A_i, places upon

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a subset, S, of some set of resources $\mathcal{R}$, is *context independent*. In other words, this value, $u_i(S)$, does not vary regardless of what allocation of the resources $\mathcal{R}/S$ is used for the other agents in the systems. It is not difficult, however, to envisage situations which such context independent interpretations of utility have difficulty modelling. Thus, in a 3 agent system, A_1 and A_2 may wish to act in partnership against A_3 in determining a partition of a resource set. In this context the value that A_1 places upon a particular subset S may vary according to which subset of the remaining resources A_2 obtains. In addition to such coalitional settings, one may wish to model situations whereby an individual agent will assess a given assignment as having greater worth if it arises in a context for which some other agent is not granted certain resources. As a more concrete example of where context dependent evaluation is significant one can consider partnership games such as Bridge, in which setting it is well-known that the 'value', in terms of trick taking potential, of a given hand may depend significantly on the distribution of the remaining cards among the other three players.

In this paper we develop an approach to the analysis of context dependent resource allocation settings, the central component of which allows agents to discriminate among different overall allocations under which it receives a particular set of resources. The basic approach is given with other definitions in Section 2. The main aim of this paper is to initiate the study of our context dependent model by considering a number of 'natural' decision questions within it. A selection of these is presented in Section 3 and related complexity classifications obtained in Section 4. Conclusions are given in the final section.

2 Definitions

The basic setting we are concerned with is encapsulated in the following definition.

Definition 1 *A* resource allocation setting *is defined by a triple* $\langle \mathcal{A}_n, \mathcal{R}_m, \mathcal{U} \rangle$ *where*

$$\mathcal{A}_n = \{A_1, A_2, \ldots, A_n\} \quad ; \quad \mathcal{R}_m = \{r_1, r_2, \ldots, r_m\}$$

are, respectively, a set of (at least two) agents and a collection of (non-shareable) resources. A utility function, *u, is a mapping from subsets of $\mathcal{R}_m$ to rational values. Each agent $A_i \in \mathcal{A}$ has associated with it a particular utility function u_i, so that $\mathcal{U}$ is $\langle u_1, u_2, \ldots, u_n \rangle$. An* allocation *$P$ of $\mathcal{R}_m$ among $\mathcal{A}_n$ is a partition $\langle P_1, P_2, \ldots, P_n \rangle$ of $\mathcal{R}_m$. We use the notation $\Pi_{n,m}$ to indicate the set of all distinct allocations of $\mathcal{R}_m$ among $\mathcal{A}_n$, noting that there are exactly n^m of these. The value $u_i(P_i)$ is called the* utility *of the resources assigned to A_i.*

The main aspect of the form of Definition 1 that we wish to address concerns its assumption that for any $S \subseteq \mathcal{R}_m$ and agent A_i, for any allocations

$$\begin{array}{rcl} P & = & \langle P_1, \ldots, P_{i-1}, S, P_{i+1}, \ldots, P_n \rangle \\ Q & = & \langle Q_1, \ldots, Q_{i-1}, S, Q_{i+1}, \ldots, Q_n \rangle \end{array}$$

The value $u_i(S)$ within the allocation P is exactly the same as its value within the allocation Q, i.e. $u_i(S)$ is invariant over all allocations of $\mathcal{R}_m/S$ among the other agents.

The basic mechanism we use to allow an agent A_i to discriminate between such allocations as P and Q above is that of a *ranking function*.

Definition 2 *A* prioritised resource allocation setting *(PRAS) is defined by a pair* $\langle \langle \mathcal{A}_n, \mathcal{R}_m, \mathcal{U} \rangle, \mathcal{V} \rangle$ *where:* $\langle \mathcal{A}_n, \mathcal{R}_m, \mathcal{U} \rangle$ *is a resource allocation setting, and* $\mathcal{V} = \langle \rho_1, \rho_2, \ldots, \rho_n \rangle$ *defines a collection of* ranking functions*. The ranking function for* A_i*,* ρ_i*, maps each* $P \in \Pi_{n,m}$ *to a non-negative integer* $\rho_i(P)$ *in the range* $[0, n^m - 1]$*. For a given allocation,* $P \in \Pi_{n,m}$*, the n-tuple of values* $\langle \rho_1(P), \rho_2(P), \ldots, \rho_n(P) \rangle$ *is called the* preference profile *of P in* $\langle \langle \mathcal{A}_n, \mathcal{R}_m, \mathcal{U} \rangle, \mathcal{V} \rangle$*. We say that an n-tuple,* $\langle k_1, k_2, \ldots, k_n \rangle$ *of non-negative integer values is an* attainable profile *if there is an allocation, P, such that* $\rho_i(P) \leq k_i$ *for each* $1 \leq i \leq n$*. If P and Q are allocations under which* $\rho_i(P) < \rho_i(Q)$ *we say that* A_i prefers the allocation P to the allocation Q.

We note that our formulation of attainable profile requires only that each agent views the allocation to be *at least* as good as the preference rank indicated by the profile: we do not require that the specified ranking value be matched exactly.

Within any prioritised resource allocation setting, there will, for each agent, be some set of allocations that the agent regards as its most preferred. One of our principal areas of interest will concern allocations that achieve the maximal preferred status with respect to arbitrary subsets (or *coalitions*) from the set of all agents.

Definition 3 *We denote by* μ_i *that value of* ρ_i *for which there is an allocation, P, under which* $\rho_i(P) = \mu_i$ *and for all other allocations,* $Q \in \Pi_{n,m}$*,* $\rho_i(Q) \geq \mu_i$*. For an agent* A_i*, any allocation, P, that satisfies* $\rho_i(P) = \mu_i$ *is said to be* optimal with respect to agent A_i in the setting $\langle \langle \mathcal{A}_n, \mathcal{R}_m, \mathcal{U} \rangle, \mathcal{V} \rangle$*. An allocation for which* $\rho_i(P) = 0$ *is said to be an* ideal allocation with respect to A_i in the setting $\langle \langle \mathcal{A}_n, \mathcal{R}_m, \mathcal{U} \rangle, \mathcal{V} \rangle$*. For a subset* $\mathcal{C} \subseteq \mathcal{A}_n$ *(or* coalition*), an allocation P is said to be a* consensus *with respect to* $\mathcal{C}$ *if for each* $A_i \in \mathcal{C}$ *it holds that* $\rho_i(P) = \mu_i$*. An allocation, P, is an* ideal consensus *for* $\mathcal{C}$ *if* $\rho_i(P) = 0$ *for every* $A_i \in \mathcal{C}$*.*

The concept of (ideal) consensus is presented in Definition 3 with respect only to the view held by a coalition regarding its members' ranking functions, i.e. the ranking assigned by agents outside the coalition is not considered. We can, additionally, introduce a notion of allocations considered with respect to *opposing coalitions*

Definition 4 *Let $\langle\langle \mathcal{A}_n, \mathcal{R}_m, \mathcal{U}\rangle, \mathcal{V}\rangle$ be a PRAS, and $\mathcal{C} \subset \mathcal{A}_n$, $\mathcal{D} \subset \mathcal{A}_n$ be disjoint non-empty sets of agents. The coalition $\mathcal{C}$* can obstruct *the coalition $\mathcal{D}$ if: there is a subset $P_{\mathcal{C}}$ of $\mathcal{R}_m$ and an allocation P' of $P_{\mathcal{C}}$ among $\mathcal{C}$ for which if Q is* any *allocation in $\Pi_{n,m}$ with $Q_i = P'_i$ for each $A_i \in \mathcal{C}$ then Q is a consensus for $\mathcal{C}$ and for every $A_j \in \mathcal{D}$, $\rho_j(Q) > \mu_j$. We refer to such a subset $P_{\mathcal{C}}$ as an* obstructive set *for $\mathcal{C}$* with respect to *$\mathcal{D}$.*

If a given coalition $\mathcal{C}$ is capable of obstructing another coalition $\mathcal{D}$ this indicates that the members of $\mathcal{C}$ could collectively acquire and distribute some subset ($P_{\mathcal{C}}$) from the overall resource set $\mathcal{R}_m$ in such a way that regardless of how the remaining resources are divided among $\mathcal{A}_n/\mathcal{C}$, the resulting allocation will be one each $A_i \in \mathcal{C}$ views as optimal, but which *no agent* in $\mathcal{D}$ will see as optimal. One consequence arising from the idea of obstructive coalitions is that during negotiations over exchange of resources between agents, a particular subset of these, S say, may acquire a certain significance in terms of the current allocation and given coalitions, $\mathcal{C}$ and $\mathcal{D}$: S may be *critical* in the sense that were $\mathcal{C}$ to acquire S then it would, together with (some subset of) its holding under P, be able to obstruct $\mathcal{D}$. Thus in such situations, it would be in the interests of $\mathcal{C}$ to acquire the missing elements of S while, similarly, agents in $\mathcal{D}$ would not only seek to prevent this, but would also have to recognise the potential for such situations to arise. Formally,

Definition 5 *Let $\langle\langle \mathcal{A}_n, \mathcal{R}_m, \mathcal{U}\rangle, \mathcal{V}\rangle$ be a PRAS, $\mathcal{C} \subset \mathcal{A}_n$, $\mathcal{D} \subset \mathcal{A}_n$ be disjoint non-empty sets of agents, and $P \in \Pi_{n,m}$ an allocation of $\mathcal{R}_m$ among $\mathcal{A}_n$. We say that the subset $S \subset \mathcal{R}_m$ is* critical *for $\langle \mathcal{C}, \mathcal{D}\rangle$* in the allocation *$P$ if: $S \not\subseteq \cup_{i \in \mathcal{C}} P_i$ and $S \cup \ \cup_{i\in\mathcal{C}} P_i$ is an obstructive set for $\mathcal{C}$ with respect to $\mathcal{D}$. More generally, if S, T are disjoint subsets of $\mathcal{R}_m$ with $T \subseteq \cup_{i\in\mathcal{C}} P_i$ and $S \not\subset \cup_{i\in\mathcal{C}} P_i$, we say that $\langle S, T\rangle$ is a* critical exchange for *$\mathcal{C}$* with respect to *$\mathcal{D}$ if $S \cup \ (\cup_{i\in\mathcal{C}} P_i)/T$ is an obstructive set for $\mathcal{C}$ with respect to $\mathcal{D}$.*

In the context of Defintion 5, it is advantageous for a coalition to identify and acquire those resources in a critical set: in such circumstances the coalitions can then achieve its most preferred allocations while preventing a select group of other agents achieving theirs. For a critical exchange, a coalition in order to reach a similar state must additionally arrange that some of its currently held resources are reassigned to other agents.

In using the concept of rank functions, Definition 2 provides one mechanism for an agent, A_i, to discriminate, should it wish to do so, between the $(n-1)^{m-|P_i|}$

distinct allocations to $\mathcal{A}_n/\{A_i\}$ that are consistent with A_i being assigned $P_i \subseteq \mathcal{R}_m$. In addition we obtain an approach that can be used to describe a number of ideas examined in earlier work. Consider, for example, the concept of an allocation being "envy-free" studied in, e.g. [8].

Definition 6 *For a resource allocation setting $\langle \mathcal{A}, \mathcal{R}, \mathcal{U} \rangle$ and an allocation P of $\mathcal{R}$ among $\mathcal{A}$, P is* envy free *if for each distinct pair, i and j, $u_i(P_i) \geq u_i(P_j)$, i.e. in an envy-free allocation, A_i values what it has been given at least as highly as it would value the resources granted to any other agent by the allocation.*

This concept is easily encapsulated within a prioritised resource allocation setting: define the ranking function $\rho_i(P)$ as,

$$\rho_i(P) \;=\; |\{\, j \; : u_i(P_j) > u_i(P_i)\}|$$

In this way an allocation, P, is envy-free if and only if $\rho_i(P) = 0$ for each i, or, in terms of Definition 3: P is envy-free if and only if P is an ideal consensus with respect to the coalition of all agents $\mathcal{A}_n$.

We note that we may recover the standard mechanism for an agent to distinguish between allocations (within a non-prioritised setting) merely by considering a decreasing order of the 2^m potential values $u_i(S)$ where $S \subseteq \mathcal{R}_m$ and fixing $\rho_i(P)$ to be the position of $u_i(P_i)$ within this, so that higher valued resource subsets are given preference.

3 Decision Problems for Prioritised Settings

Our aim in this preliminary study is to consider prioritised resource allocation settings with respect to complexity issues. Thus, we now present and discuss a number of decision problems that naturally arise in this model. In Section 4 we then obtain results regarding their computational complexity.

The first set of such problems addresses questions concerning to what extent a given allocation can be improved.

Definition 7 *The decision problem* Subjective Improvement *(*SI*) takes as an instance a PRAS, $\langle\langle \mathcal{A}_n, \mathcal{R}_m, \mathcal{U}\rangle, \mathcal{V}\rangle$, an allocation $P \in \Pi_{n,m}$ and an index i with $1 \leq i \leq n$. The instance is accepted if there is an allocation $Q \in \Pi_{n,m}$ for which $\rho_i(Q) < \rho_i(P)$.*

The decision problem Objective Improvement *(*OI*) takes as an instance a PRAS, $\langle\langle \mathcal{A}_n, \mathcal{R}_m, \mathcal{U}\rangle, \mathcal{V}\rangle$ and an allocation $P \in \Pi_{n,m}$ with the instance accepted if there is an allocation $Q \in \Pi_{n,m}$ for which $\wedge_{i=1}^{n}(\rho_i(Q) < \rho_i(P))$ holds true.*

The decision problem Pareto Optimality *(*PO*) takes as an instance a PRAS,* $\langle\langle\mathcal{A}_n,\mathcal{R}_m,\mathcal{U}\rangle,\mathcal{V}\rangle$ *and an allocation* $P \in \Pi_{n,m}$ *with the instance accepted if for every allocation* $Q \in \Pi_{n,m}$*: should* $\rho_i(Q) < \rho_i(P)$ *for some* $1 \leq i \leq n$ *then* $\rho_j(Q) > \rho_j(P)$ *for some* $1 \leq j \leq n$*.*

Subjective Improvement deals with whether a single specified agent can realise an allocation that it prefers to the given one, whereas the focus of Objective Improvement is on allocations which improve the preferences of all agents in the system. The concept of Pareto Optimality has, of course, received much attention both within the field of coalitional game theory and multiagent resource allocation models. Our description of this in terms of ranking profiles, subsumes the latter class of applications, so that in informal terms, an allocation is Pareto Optimal if some agent can achieve a more preferred allocation only at the cost of another agent being penalised.

Definition 8 *An instance of the decision problem* Attainable Profile *(*AP*) comprises a PRAS,* $\langle\langle\mathcal{A}_n,\mathcal{R}_m,\mathcal{U}\rangle,\mathcal{V}\rangle$ *and an n-tuple* $\langle k_1, k_2, \ldots, k_n\rangle$ *of non-negative integers. The instance is accepted if the profile* $\langle k_1, k_2, \ldots, k_n\rangle$ *is attainable in* $\langle\langle\mathcal{A}_n,\mathcal{R}_m,\mathcal{U}\rangle,\mathcal{V}\rangle$*.*

Definition 9 *An instance of the decision problem* Obstructive Coalition *(*OC*) comprises a PRAS,* $\langle\langle\mathcal{A}_n,\mathcal{R}_m,\mathcal{U}\rangle,\mathcal{V}\rangle$ *and two disjoint subsets* $\mathcal{C}$*,* $\mathcal{D}$ *from* $\mathcal{A}_n$*. The instance is accepted if the coalition* $\mathcal{C}$ *can obstruct the coalition* $\mathcal{D}$ *in the setting* $\langle\langle\mathcal{A}_n,\mathcal{R}_m,\mathcal{U}\rangle,\mathcal{V}\rangle$*.*

An instance of the decision problem Critical Set *(*CS*) consists of a PRAS,* $\langle\langle\mathcal{A}_n,\mathcal{R}_m,\mathcal{U}\rangle,\mathcal{V}\rangle$*, two disjoint non-empty subsets* $\mathcal{C}$*,* $\mathcal{D}$ *from* $\mathcal{A}_n$*, and an allocation* $P \in \Pi_{n,m}$ *for which* $\rho_i(P) > \mu_i$ *for each* $i \in \mathcal{C} \cup \mathcal{D}$*. An instance is accepted if: there is a set* $S \subset \mathcal{R}_m$ *such that S is critical for* $\langle\mathcal{C},\mathcal{D}\rangle$ *in the allocation P.*

We note the condition $\rho_i(P) > \mu_i$ for each $i \in \mathcal{C} \cup \mathcal{D}$ implies that each agent involved in the two coalitions has a reason to seek out some exchange in the resources held in the hope that it will reach a position it regards as optimal.

4 Complexity in Prioritised Settings

Before presenting our complexity results, we deal with one technical issue concerning the representation of instances of prioritised resource allocation settings. We recall that two elements of these are the *n*-tuple of utility functions, $\mathcal{U} = \langle u_1, \ldots, u_n\rangle$ each of which maps subsets of $\mathcal{R}_m$ to rational values; and the *n*-tuple of ranking functions $\mathcal{V} = \langle \rho_1, \ldots, \rho_n\rangle$ each of which maps allocations in $\Pi_{n,m}$ to non-negative integer values. In giving representations of these in instances, say, of AP and OC we face the problem that the domains are exponentially large in the value m: 2^m subsets of $\mathcal{R}_m$, n^m allocations in $\Pi_{n,m}$. Thus were these to be presented by explicitly enumerating pairs of subset and value (for $\mathcal{U}$) or pairs of allocation and value (for $\mathcal{V}$) the space taken by the instance encoding would be unreasonable for all but modest values of m. It is unlikely to be the case that such enumerative descriptions would be used in practice, and it could also happen that assessments of the problem complexity in terms of the instance size would result in apparent 'polynomial' time algorithms: such methods would, however, be polynomial only by virtue of the infeasible nature of the instance representation. In order to circumvent these difficulties it is necessary to adopt a convention for describing $\mathcal{U}$ and $\mathcal{V}$ that will allow, in such cases were it is possible to do so, these elements to be described by an encoding whose length is polynomial in $n+m$ and which allows the relevant functions to be evaluated efficiently (in terms of the encoding size). While there are a wide range of possible schemes that could be employed, the one which will be assumed in our subsequent development is the following: each utility function u_i is presented by a *combinational logic network* over the basis of 2-input Boolean functions (i.e. a straight-line program) with m input bits and t_i output bits, this network having the property that for each $S \subseteq \mathcal{R}_m$ if the inputs are instantiated by the Boolean value $(r_i \in S)$ then $val(S)$ the t_i-bit binary value induced at the output will be such that $u_i(S) = val(S)/m$. We employ a similar formalism for encoding ρ_i, this time employing a network with nm Boolean inputs to encode allocations in $\Pi_{n,m}$. For further details on this widely-studied model of function computation we refer the reader to any of the standard monographs such as [4].

Theorem 1 AP *is* NP*–complete.*

Proof. Membership NP is immediate from the non-deterministic algorithm that simply guesses an allocation $P \in \Pi_{n,m}$ and checks that $\wedge_{i=1}^{n}(\rho_i(P) \leq k_i)$ holds. Recalling our convention for representing $\mathcal{V}$ each test $\rho_i(P) \leq k_i$ can be performed in time polynomial (in fact, linear) in the length of the encoding of ρ_i.

For NP–hardness we will, in fact, prove a rather stronger result: AP is NP–hard even when instances are restricted to 2 agents, with the profile $\langle k_1, k_2 \rangle = \langle 0, 0 \rangle$ and for a fixed pair of ranking functions $\langle \rho_1, \rho_2 \rangle$ whose definition, given an allocation $P = \langle P_1, P_2 \rangle$, is

$$\rho_1(P) = \begin{cases} 1 & \text{if} \quad u_1(P_2) > u_1(P_1) \\ 0 & \text{if} \quad u_1(P_2) \leq u_1(P_1) \end{cases}$$

$$\rho_2(P) = \begin{cases} 1 & \text{if} \quad u_2(P_1) > u_2(P_2) \\ 0 & \text{if} \quad u_2(P_1) \leq u_2(P_2) \end{cases}$$

We employ a reduction from 3-SAT, in which instances $\Phi(Z_t) = \wedge_{i=1}^{r}(y_{i,1} \vee y_{i,2} \vee y_{i,3})$, are limited to those for which the number of propositional variables, t, is *odd*. It is a trivial matter to show that this restriction makes no difference to the NP-complete status of 3-SAT. Given such an instance, and the fact that $n = 2$, $\langle k_1, k_2\rangle = \langle 0, 0\rangle$ with $\langle \rho_1, \rho_2\rangle$ as defined above, we need only specify $\mathcal{R}$ and the utility functions $\langle u_1, u_2\rangle$ to be employed. Fix $\mathcal{R} = \{z_1, z_2, \ldots, z_t\}$, i.e. the resource being divided is the set of propositional variables defining $\Phi(Z_t)$. For $W \subseteq \mathcal{R}$, the instantiation $pos(W)$ is given by $z_i = \top$ if $z_i \in W$, $z_i = \bot$ if $z_i \notin W$; similarly, the instantiation $neg(W)$ is given by $pos(\mathcal{R}/W)$. The utility functions, $\langle u_1, u_2\rangle$ are specified by,

$$u_1(S) = \begin{cases} 2t & \text{if} \quad \Phi(pos(S)) = \top \\ |S| & \text{if} \quad \Phi(pos(S)) \neq \top \end{cases}$$

$$u_2(S) = \begin{cases} 2t & \text{if} \quad \Phi(neg(S)) = \top \\ |S| & \text{if} \quad \Phi(neg(S)) \neq \top \end{cases}$$

We claim that the profile $\langle 0, 0\rangle$ is attainable in $\langle\langle \mathcal{A}_2, \mathcal{R}, \langle u_1, u_2\rangle\rangle, \langle \rho_1, \rho_2\rangle\rangle$ if and only if $\Phi(Z_t)$ is satisfiable. Suppose first that $\alpha = \langle a_1, \ldots, a_t\rangle$ is an instantiation of Z_t that satisfies Φ. Consider the subset W_α for which $z_i \in W_\alpha$ if and only if $a_i = \top$. For this, $u_1(W_\alpha) = u_2(\mathcal{R}/W_\alpha) = 2t$, and since $2t$ is the maximum value u_i can attain, it follows that $\rho_1(\langle W_\alpha, \mathcal{R}/W_\alpha\rangle) = \rho_2(\langle W_\alpha, \mathcal{R}/W_\alpha\rangle) = 0$ as required. On the other hand, suppose that $\langle S, \mathcal{R}/S\rangle$ is an allocation which attains the profile $\langle 0, 0\rangle$. Consider the instantiation $pos(S)$ and suppose that this does *not* satisfy $\Phi(Z_t)$. In this case we have, $u_1(S) = |S|$ and $u_1(\mathcal{R}/S) \in \{t - |S|, 2t\}$; similarly $u_2(\mathcal{R}/S) = t - |S|$ and $u_2(S) \in \{|S|, 2t\}$. If $u_1(\mathcal{R}/S) = 2t$ and $u_2(S) = 2t$ then the profile of $\langle S, \mathcal{R}/S\rangle$ is $\langle 1, 1\rangle$ – contradicting the assumption that $\langle 0, 0\rangle$ has been attained. Noting that $u_1(\mathcal{R}/S) = 2t$ if and only if $u_2(S) = 2t$ this leaves only the cases $u_1(S) = |S|$, $u_1(\mathcal{R}/S) = t - |S|$, $u_2(\mathcal{R}/S) = t - |S|$, $u_2(S) = |S|$. Now we recall that t is *odd* and therefore we cannot have $|S| = t - |S|$. We thus obtain the contradiction should $pos(S)$ fail to satisfy $\Phi(Z_t)$ that the profile $\langle \rho_1(S), \rho_2(\mathcal{R}/S)\rangle \in \{\langle 0, 1\rangle, \langle 1, 0\rangle, \langle 1, 1\rangle\}$. It follows that $\Phi(Z_t)$ is satisfiable if and only the profile $\langle 0, 0\rangle$ is attainable in the constructed setting. ▫

The following Corollaries of Theorem 1 are easily proved, following directly from the reduction used or by minor modifications to it.

Corollary 1 *The following all hold even when $|\mathcal{A}| = 2$:*

a. SI *is* NP*–complete.*

b. OI *is* NP*–complete.*

c. PO *is* CO-NP*–complete.*

d. *Deciding if a resource allocation setting* $\langle \mathcal{A}_n, \mathcal{R}_m, \mathcal{U} \rangle$ *admits an* envy-free *allocation is* NP*–complete.*

e. *Deciding if an agent within a PRAS has an ideal allocation is* NP*–complete.*

f. *Deciding if a PRAS has an ideal consensus with respect to* $\mathcal{A}_n$ *is* NP*–complete.*

Proof. Omitted. ▫

Theorem 1 and its corollaries, although couched in terms of prioritised settings, have analogous phrasings in the standard setting of Definition 1. Our remaining results, however, consider questions which have no natural counterpart in this arena, i.e. they arise specifically in the treatment of context dependent schemes. We first consider determining whether a coalition has the capability to obstruct another by identifying an obstructive set of resources.

Theorem 2 OC *is* Σ_2^p*–complete.*

Proof. For membership in Σ_2^p it suffices to observe that $\langle\langle\langle \mathcal{A}_n, \mathcal{R}_m, \mathcal{U}\rangle, \mathcal{V}\rangle, \mathcal{C}, \mathcal{D}\rangle$ is accepted as an instance of OC if and only if: $\exists \langle P_{i_1}, \ldots, P_{i_k}\rangle \; \forall \, Q \in \Pi_{n,m}$, should it be the case that $\wedge_{i_j \in \mathcal{C}} P_{i_j} = Q_{i_j}$ then

$$\left(\bigwedge_{i_j \in \mathcal{C}} \rho_{i_j}(Q) = \mu_{i_j} \wedge \bigwedge_{i_k \in \mathcal{D}} \rho_{i_k}(Q) > \mu_{i_k} \right)$$

To establish that OC is Σ_2^p–hard, we give a reduction from QSAT_2^{Σ} instances of which comprise a CNF formula $\Phi(X_t, Y_t)$ defined over 2 disjoint sets of propositional variables. An instance of QSAT_2^{Σ} is accepted if there is an instantion, α_X of X_t under which for all instantiations, β_Y of Y_t we have $\Phi(\alpha_X, \beta_Y) = \perp$.

Given $\Phi(X_t, Y_t)$ we construct the following instance of OC.

The set of agents contains three members $\{A_1, A_2, A_3\}$ while $\mathcal{R}$ contains the $2t$ elements $\{x_1, \ldots, x_t, y_1, \ldots, y_t\}$. We may fix $u_i(S) = 0$ for each i, since we only need to define each ranking function appropriately. This we do as follows. Using X_i and Y_i to denote the sets $P_i \cap \{x_1, \ldots, x_t\}$ and $P_i \cap \{y_1, \ldots, y_t\}$, for

$P = \langle P_1, P_2, P_3 \rangle \in \Pi_{3,2t}$

$$\rho_1(P) = \begin{cases} 0 & \text{if } Y_1 = \emptyset \text{ and } \Phi(pos(X_1), pos(Y_2)) = \bot \\ & \text{and } \Phi(pos(X_1), pos(Y_3)) = \bot \\ 1 & \text{otherwise} \end{cases}$$

$$\rho_2(P) = \begin{cases} 0 & \text{if } Y_1 \neq \emptyset \text{ or } \Phi(pos(X_1), pos(Y_2)) = \top \\ 1 & \text{otherwise} \end{cases}$$

$$\rho_3(P) = \begin{cases} 0 & \text{if } Y_1 \neq \emptyset \text{ or } \Phi(pos(X_1), pos(Y_3)) = \top \\ 1 & \text{otherwise} \end{cases}$$

where $pos(S)$ is are as defined in the proof of Theorem 1 with respect to the sets X_t and Y_t. We complete the construction of the instance by setting $\mathcal{C} = \{A_1\}$ and $\mathcal{D} = \{A_2, A_3\}$.

We claim that our construction is accepted as an instance of OC if and only if $\Phi(X_t, Y_t)$ is accepted as an instance of QSAT$_2^\Sigma$. Suppose that $\Phi(X_t, Y_t)$ is a positive instance of QSAT$_2^\Sigma$ and let $\alpha_X = \langle a_1, a_2, \ldots, a_t \rangle$ be the witnessing instantiation of X_t. Consider the allocation $P_1 = \{x_i : a_i = \top\}$. Then for any allocation, $Q \in \Pi_{3,2t}$ for which $Q_1 = P_1$ we have $\rho_1(Q) = 0$ since Y_t is distributed among A_2 and A_3 and, from the definition of α_X, we get

$$\Phi(\alpha_X, pos(Y_2)) = \Phi(\alpha_X, pos(Y_3)) = \bot$$

and thus, in addition, $\rho_2(Q) = \rho_3(Q) = 1 > \mu_2 = \mu_3 = 0$. We note that the condition on Y_1 means that there are allocations for which $\rho_2(Q) = \rho_3(Q) = 0$. We deduce that $\langle\langle\langle \mathcal{A}_3, \mathcal{R}, \mathcal{U}\rangle, \mathcal{V}\rangle, \{A_1\}, \{A_2, A_3\}\rangle$ is accepted as an instance of OC.

On the other hand suppose the constructed instance is a positive instance of OC and consider a subset P_1 of $\mathcal{R}$ that witnesses this. It is certainly the case that $Y_1 = \emptyset$. It must also hold, however, that for every allocation, Q, with $Q_1 = P_1$ and $\{y_1, \ldots, y_t\}$ distributed over $\langle Q_2, Q_3 \rangle$ we have $\rho_1(Q) = 0$ and $\rho_2(Q) = \rho_3(Q) = 1$, i.e. $\Phi(pos(X_1), pos(Y_2)) = \Phi(pos(X_1), pos(Y_3)) = \bot$. We deduce that the instantiation $pos(X_1)$ witnesses $\Phi(X_t, Y_t)$ as a positive instance of QSAT$_2^\Sigma$ completing the proof that OC is Σ_2^p–complete. □

We obtain a similar classification for the decision problem Critical Set.

Theorem 3 CS *is* Σ_2^p*–complete.*

Proof. Membership in Σ_2^p follows by the algorithm which on a given instance $\langle\langle\langle \mathcal{A}_n, \mathcal{R}_m, \mathcal{U}\rangle, \mathcal{V}\rangle, \mathcal{C}, \mathcal{D}, P\rangle$, guesses a subset, S, of $\cup_{i \notin \mathcal{C}} P_i$ together with an allocation, $\langle P_{i_1}, \ldots, P_{i_r}\rangle$ of $S \cup \cup_{i \in \mathcal{C}} P_i$ among $\mathcal{C}$ (noting that these can be combined in a single Σ_1^p, i.e. NP, stage), followed by checking for each $Q \in \Pi_{n,m}$ that should $Q_{i_j} = P_{i_j}$ for each $1 \leq j \leq r$, then $\rho_i(Q) = \mu_i$ for each $A_i \in \mathcal{C}$, while $\rho_j(Q) > \mu_j$ for each $j \in \mathcal{D}$. The entire process is easily accomplished by a Σ_2^p algorithm.

For Σ_2^p–hardness we again employ a reduction from QSAT$_2^\Sigma$. Given an instance $\Phi(X_t, Y_t)$ of this we form an instance of CS in which

$$\begin{array}{lcl} \mathcal{A} & = & \{A_1, A_2, A_3, A_4\} \\ \mathcal{R} & = & \{x_1, \ldots, x_t, y_1, \ldots, y_t\} \\ \mathcal{C} & = & \{A_1, A_2\} \\ \mathcal{D} & = & \{A_3\} \\ \langle P_1, P_2, P_3, P_4 \rangle & = & \langle \emptyset, \emptyset, \emptyset, \mathcal{R} \rangle \end{array}$$

We again may use $u_i(S) = 0$ for each agent. It remains to define $\mathcal{V} = \langle \rho_1, \rho_2, \rho_3, \rho_4 \rangle$. For $Q \in \Pi_{4,2t}$, $\rho_i(Q)$ is defined as below, where, as in Theorem 2, the notations X_i and Y_i are used for $P_i \cap \{x_1, \ldots, x_t\}$ and $P_i \cap \{y_1, \ldots, y_t\}$.

$$\rho_1(Q) = \begin{cases} 0 & \text{if } Y_1 = \emptyset \text{ and } X_1 \cup X_2 = \{x_1, \ldots, x_t\} \\ & \text{and } \Phi(pos(X_1), pos(Y_3)) = \bot \\ 1 & \text{if } X_1 \cup X_2 \subset \{x_1, \ldots, x_t\} \text{ and } Y_1 = \emptyset \\ 2 & \text{otherwise} \end{cases}$$

$$\rho_2(Q) = \begin{cases} 0 & \text{if } Y_2 = \emptyset \text{ and } X_1 \cup X_2 = \{x_1, \ldots, x_t\} \\ & \text{and } \Phi(neg(X_2), neg(Y_4)) = \bot \\ 1 & \text{if } X_1 \cup X_2 \subset \{x_1, \ldots, x_t\} \text{ and } Y_2 = \emptyset \\ 2 & \text{otherwise} \end{cases}$$

$$\rho_3(Q) = \begin{cases} 0 & \text{if } Y_1 = \emptyset \text{ and } X_3 \cup X_4 = \emptyset \\ & \text{and } \Phi(pos(X_1), pos(Y_3)) = \top \\ 1 & \text{if } X_3 \neq \emptyset \text{ or } X_4 \neq \emptyset \\ 2 & \text{otherwise} \end{cases}$$

$$\rho_4(Q) = \begin{cases} 0 & \text{if } Y_2 = \emptyset \text{ and } X_3 \cup X_4 = \emptyset \\ & \text{and } \Phi(neg(X_2), neg(Y_4)) = \top \\ 1 & \text{otherwise} \end{cases}$$

For the allocation

$$P = \langle \emptyset, \emptyset, \emptyset, \mathcal{R} \rangle$$

we get

$$\langle \rho_1(P), \rho_2(P), \rho_3(P), \rho_4(P) \rangle = \langle 1, 1, 1, 1 \rangle$$

Now suppose that $\Phi(X_t, Y_t)$ is accepted as an instance of QSAT$_2^\Sigma$, letting $\alpha_X = \langle a_1, \ldots, a_t \rangle$ be an instantiation witnessing this. Choosing S to be the set containing exactly those x_i for which $a_i = \top$, gives a critical set for $\langle\{A_1, A_2\}, \{A_3\}\rangle$ with respect to the allocation $\langle\emptyset, \emptyset, \emptyset, \mathcal{R}\rangle$, simply by assigning $Q_1 = S$, $Q_2 = X_t/S$. Then for any allocation $\langle Q_3, Q_4\rangle$ of Y_t among $\{A_3, A_4\}$ we get,

$$\rho_1(\langle Q_1, Q_2, Q_3, Q_4\rangle) = \rho_2(Q_1, Q_2, Q_3, Q_4) = 0$$

and $\rho_3(\langle Q_1, Q_2, Q_3, Q_4\rangle) = 2$.

On the other hand, suppose that our construction yields a positive instance of CS and consider the sets S_1, S_2 allocated to $\{A_1, A_2\}$ from the critical set S witnessing acceptance. It must certainly hold that $S = \{x_1, \ldots, x_t\}$ for any other choice will result in $\rho_1(Q) > 0$ and $\rho_2(Q) > 0$ when $Q_1 = S_1$ and $Q_2 = S_2$. It follows that the resource subset $\{y_1, \ldots, y_t\}$ is distributed between $\{A_3, A_4\}$ and no matter how such a distribution is made, we get $\rho_3(Q) = 2$ and $\rho_4(Q) = 1$ (since $X_3 = X_4 = \emptyset$). From this we see that, regardless of whichever choice of Y_3 is made, we have $\Phi(pos(X_1), pos(Y_3)) = \bot$ and deduce that $\Phi(X_t, Y_t)$ is accepted as an instance of QSAT$_2^\Sigma$ by virtue of the instantiation $pos(X_1)$ of X_t. This completes the proof that CS is Σ_2^p–complete. □

The 4 agent setting employed in the proof of Theorem 3 raises a number of issues of interest. Not least among these is the question of optimal strategies for the two coalitions - $\{A_1, A_2\}$ and $\{A_3\}$. Given an arbitrary formula instance $\Phi(X_t, Y_t)$ the resource set in the proof construction is, at first, held in its entirety by A_4. We may observe that A_4, since it is a member of neither coalition, could be regarded as neutral to the interests of both, even though it can only achieve a most preferred allocation if there are circumstances which allow A_3 to do the same. As such we might assume that A_4 has no reason to object to subsets of its orginal holding being claimed by any of the other three agents: such largesse cannot render the resulting allocation *less* preferred than its initial one. Consider now the situation that the coalition $\{A_1, A_2\}$ faces: certainly it can never be to its advantage for either member to obtain elements of $\{y_1, \ldots, y_t\}$. Thus the decision facing this coalition is whether to obtain *all* of the resources in $\{x_1, \ldots, x_t\}$ to distribute among its members or whether only to acquire a *proper subset* of these: the first could allow $\{A_1, A_2\}$ to obstruct $\{A_3\}$ but may also let A_3 devise allocations that are found *worse* than the initial one. On the other hand, the second choice prevents an ideal state but can never leave $\{A_1, A_2\}$ less content. Similar considerations impinge upon the choices made by A_3: if it elects to obtain some element of $\{x_1, \ldots, x_t\}$ and retain it then it does so at the cost of never being able to achieve an ideal allocation even if it *were possible* to do so. If, however, A_3 avoids choosing any element of $\{x_1, \ldots, x_t\}$, there is a risk that A_3 ends up worse off.

5 Conclusions

The principal contention of this paper is that the oft employed model for considered in the study of multiagent resource allocation is insufficiently expressive to address arenas wherein the worth a single agent attributes to its alloted resource is dependent on external factors. We have argued that importing a simple ranking mechanism into the standard setting provides an approach flexible enough to model such context dependent issues, illustrating this view with refererence to a select number of natural decision questions whose computational complexity has been classified. These include both problems that encompass related questions in the standard setting, e.g. Subjective Improvement, as well as a number that arise specifically in our prioritised variant, e.g. Obstructive Coalition.

Although we have chosen to present this model from the viewpoint of multiagent resource allocation and evaluation, we note that the issues motivating it are also of great relevance to more general concerns arising from scenarios modelled through some underlying set ($\mathcal{R}$) divided among a finite set of participants ($\mathcal{A}$). Thus if $\mathcal{R}$ is interpreted as a collection of beliefs, attitudes, and facts held by members of $\mathcal{A}$ then we have a framework for considering persuasive argument, e.g. in the scheme of [10], where the force and acceptance of particular claims by one agent depends not only on its own beliefs and attitudes but also on how these relate to the views endorsed by other agents. Since, in principle abstract models of argument and reasoning such as that of Dung [7] could be embedded within a multi-party debate setting, the development of these to describe relative notions of value preferences that has been initiated in the work of Bench-Capon [1, 2] may be defined through our prioritised model.

Finally we note the potentially rich seam of problems that arise in formulating strategies for coalitions to identify consensus allocations, critical sets, and obstructive possibilities. As we have outlined in the postlude to Theorem 3, even a setting comprising only 4 agents yields non-trivial strategic questions for the coalitions involved when these seek either to improve their preference or avoid it degrading.

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