Temporal Graphs: Algorithms and Complexity

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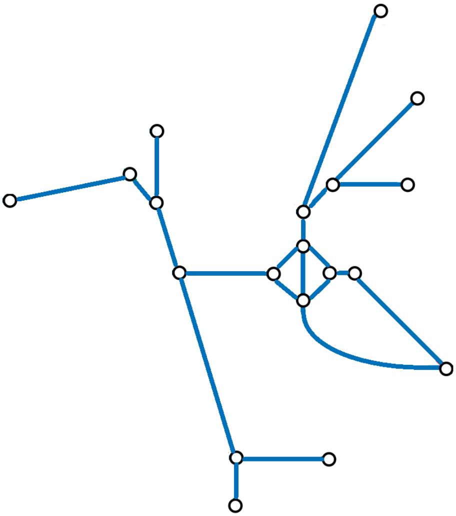
Static and Temporal Graphs



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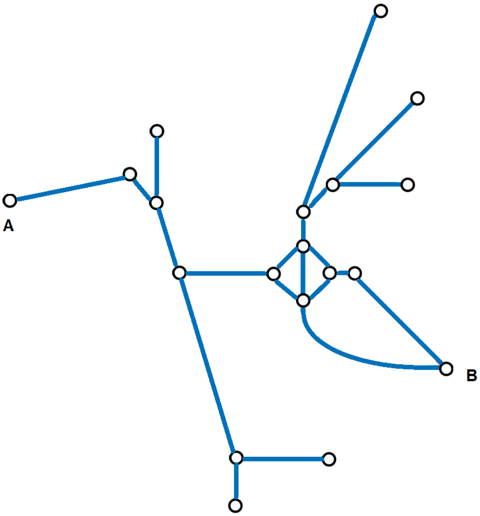

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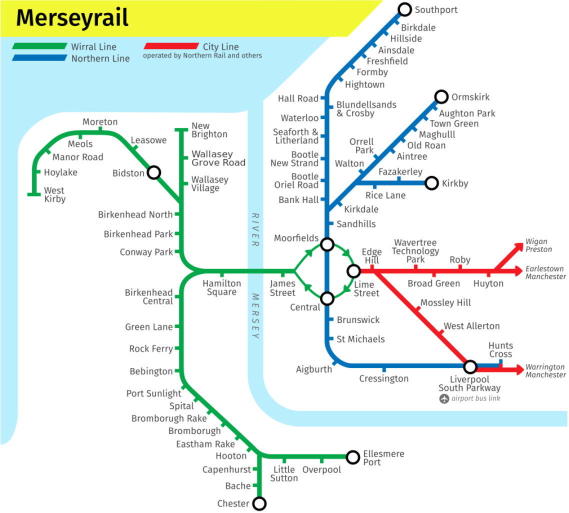

Static and Temporal Graphs

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- abstracted as graphs
- vertex $\longleftrightarrow$ elementary system unit
- edge $\longleftrightarrow$ some kind of interaction between units

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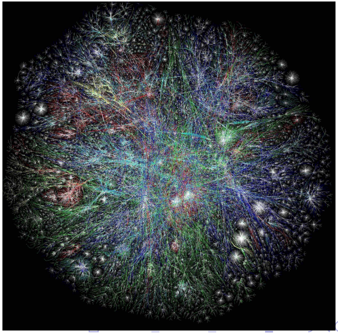
However many modern systems are highly dynamic:

- Modern communication networks, e.g., mobile ad hoc, sensor, peer-to-peer, opportunistic, delay-tolerant networks: links change dynamically at a high rate
- Social networks: friendships are added/removed, individuals leave, new ones enter
- Physical systems: e.g. systems of interacting particles
- Transportation networks: transportation units change with time their position in the network



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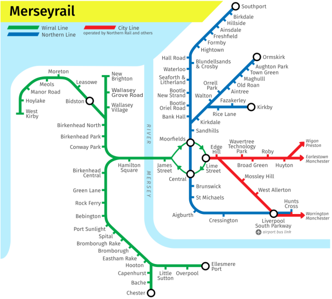
The common characteristic in all these applications:

- the graph topology is subject to discrete changes over time

$\Rightarrow$ the notion of vertex adjacency must be appropriately re-defined
(by introducing the time dimension in the graph definition)



Various graph concepts (e.g. reachability, connectivity):

- crucially depend on the exact temporal ordering of the edges

Overview

- Temporal graphs
- Temporal paths (journeys)
- Strongly connected components
- Temporal exploration
- Temporal design problems
- Stochastic temporal graphs
- Future research directions

Temporal graphs

Formally:

Definition (Temporal Graph)

A temporal graph is a pair (G, λ) where:

- $G = (V, E)$ is an underlying (di)graph and
- $\lambda : E \to 2^{\mathbb{N}}$ is a discrete time-labeling function.

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- If $t \in \lambda(e)$ then edge e is available at time t
- This formal definition (for single-availabilities per edge) embarks from:
  [Kempe, Kleinberg, Kumar, *STOC*, 2000]
  [Berman, *Networks*, 1996]
- In general every edge can have multiple availabilities
  [Mertzios, Michail, Chatzigiannakis, Spirakis, *ICALP*, 2013]

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Remarks:

- Denote by λ_{min} (resp. λ_{max}) the smallest (resp. largest) time-label in (G, λ)
  - λ_{max} may be infinite (e.g. in periodic temporal graphs)
  - If $\lambda_{\text{max}} \neq \infty$, then the age of (G, λ) is $\alpha(\lambda) = \lambda_{\text{max}} - \lambda_{\text{min}} + 1$
- Unless otherwise specified:
  - the labels are given explicitly with the input
  - $c(\lambda)$ is the total number of labels

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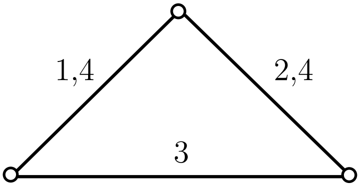

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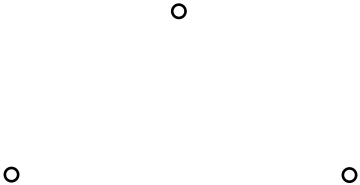

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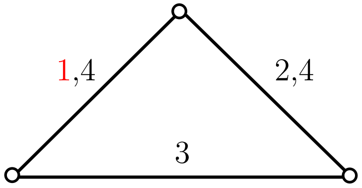

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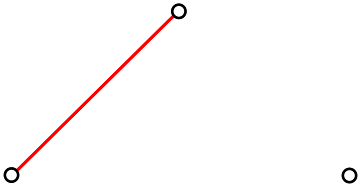

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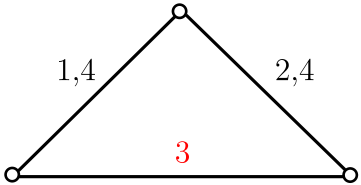

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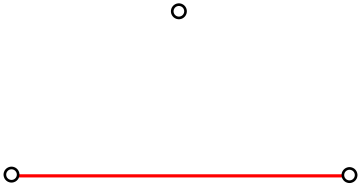

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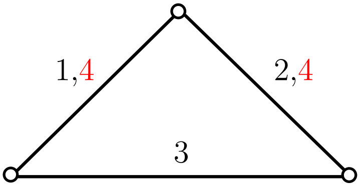

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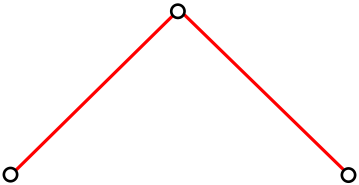

Related notions of dynamicity in graphs:

- flows over time
  [Fleischer, Skutella, *SIAM J. on Computing*, 2007]
  [Hoppe, Tardos, *Math. Oper. Res.*, 2000]
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    - continuous availabilities; natural model, different techniques
- minimum label graph problems
  [Fellows, Guo, Kanj, *J. Comp. Syst. Sci.*, 2010]
    - input: static topology G with a label on each edge, graph property Π
    - goal: find an edge subset with the smallest number of distinct labels which satisfies Π

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- dynamic graphs
  [Demetrescu, Finocchi, Italiano, *Handbook Data Str. and Appl.*, 2004]
    - topology changes via insertion/deletion of vertices/edges
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In contrast, in the context of temporal networks:

- topology is expected to change frequently and massively
  $\Rightarrow$ changes are not anomalies or exceptions
- they are rather an integral part of the system
  $\Rightarrow$ can not be reasonably modeled with network faults /failures

Temporal graphs

Temporal graphs were studied under various different names:

- time-varying graphs
  [Aaron et al., *WG*, 2014]
  [Flocchini et al., *ISAAC*, 2009]
  [Tang et al., *ACM Comp. Comm. Review*, 2010]
- evolving graphs (usually "graph-centric")
  [Avin et al., *ICALP*, 2008]
  [Clementi et al., *SIAM J. Discr. Math.*, 2010]
  [Ferreira, *IEEE Network*, 2004]
- dynamic graphs
  [Giakkoupis et al., *ICALP*, 2014]
  [Casteigts et al., *Int. J. Par., Emergent & Distr. Syst*, 2012]
  [Bhadra and Ferreira, *J. Internet Serv. Appl.*, 2012]
- graphs over time
  [Leskovec et al., *ACM Trans. Knowl. Disc. from Data*, 2007]

Temporal graphs

Recent surveys and books:

- Time-Varying Graphs and Dynamic Networks
  [Casteigts et al., *Int. J. Par., Emergent & Distr. Syst*, 2012]
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- Temporal Networks
  [Holme, Saramäki, eds., *Springer*, 2013]
  [Michail, Spirakis, *Commun. ACM*, 2018]

Overview

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- Strongly connected components
- Temporal exploration
- Temporal design problems
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- Future research directions

Temporal paths

The conceptual shift from static to temporal graphs significantly impacts:

- the definition of basic graph parameters
- the type of tasks to be computed

Graph properties can be classified as:

- a-temporal, i.e. satisfied at every instance
  - connectivity at every point in time
- temporal, i.e. satisfied over time
  - communication routes over time

Temporal paths

Definition (Temporal path; Time-respecting path; Journey)

Let (G, λ) be a temporal graph and $P = (e_1, e_2, \ldots, e_k)$ be a walk in G. A temporal path is a sequence $((e_1, \ell_1), (e_2, \ell_2), \ldots, (e_k, \ell_k))$, where:

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Causality in information dissemination:

- information "flows" along edges whose labels respect time ordering
  $\Rightarrow$ strictly increasing labels along the path
- a "static path" given "in pieces"

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Most identified temporal graph parameters are "temporal path"-related:

- temporal versions of distance, diameter, connectivity, reachability, exploration, etc.

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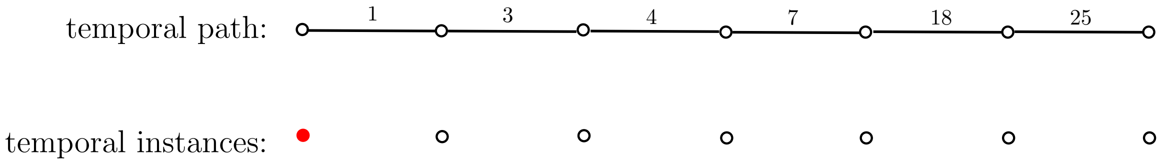

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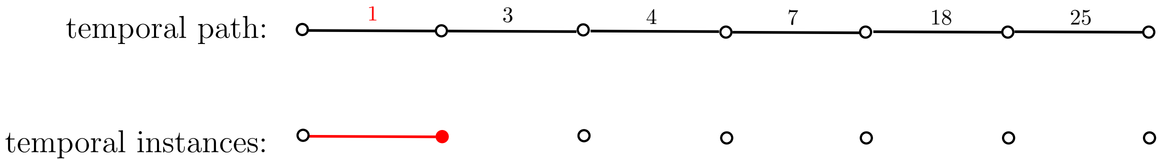

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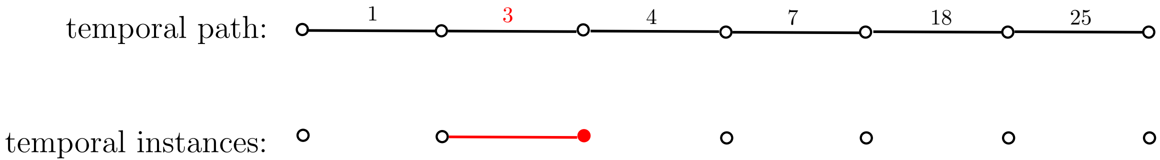

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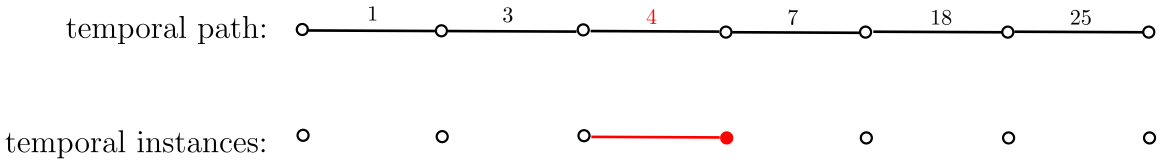

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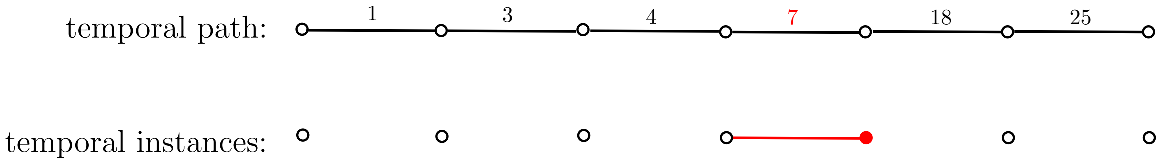

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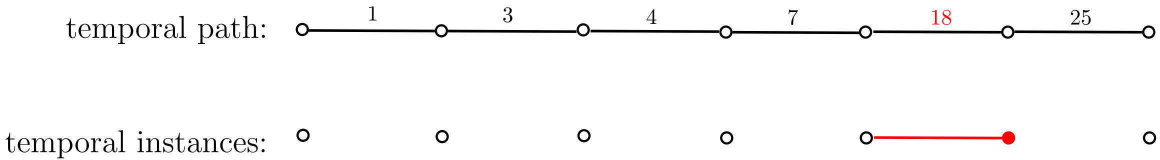

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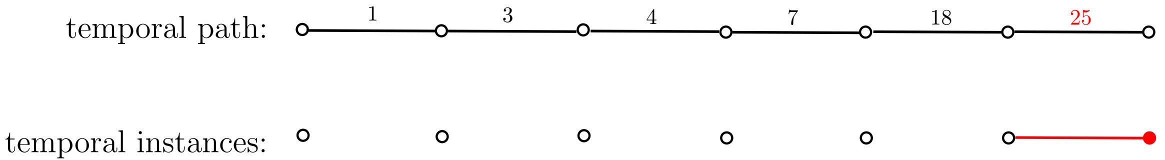

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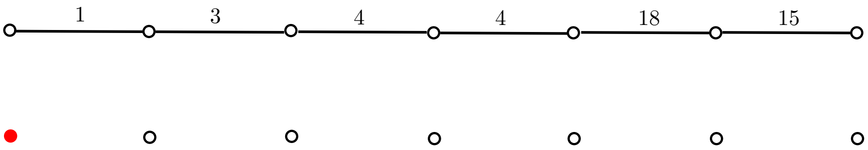

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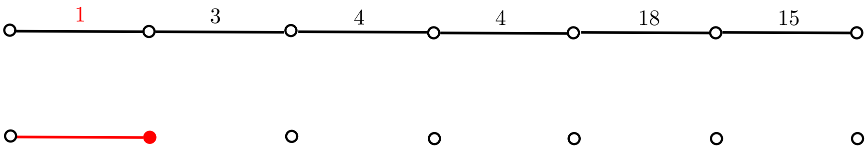

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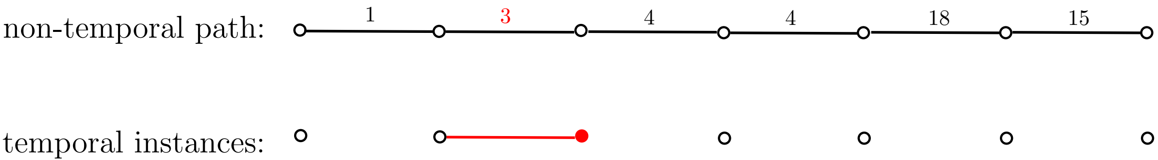

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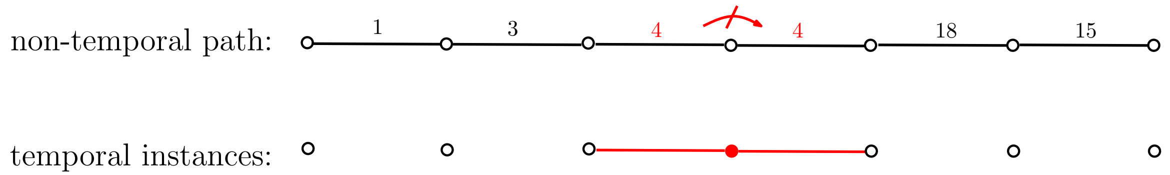

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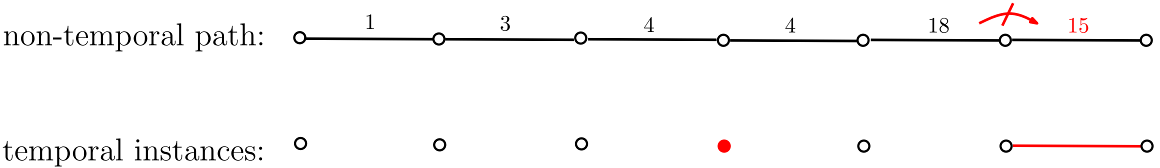

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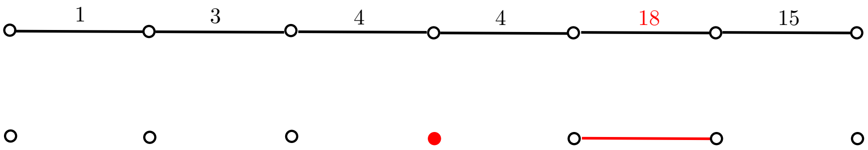

Metrics to optimize

Question: What is the temporal analogue of an s-t shortest path?

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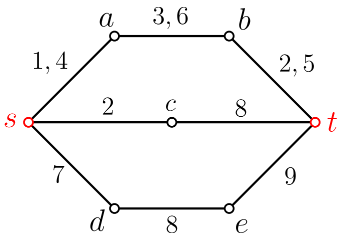

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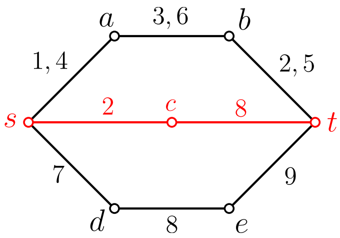

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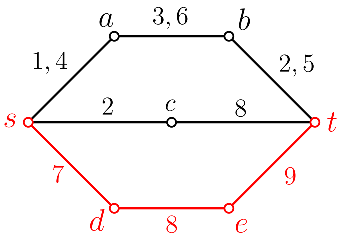

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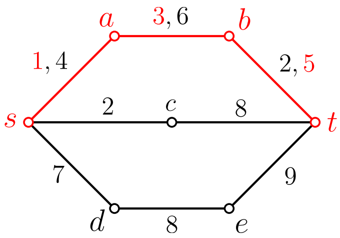

shortest: s-c-t (two edges)

fastest: s-d-e-t (no intermediate waiting)

foremost: $s - a - b - t$ (arriving at time 5)

Metrics to optimize

An easy algorithm for computing all foremost paths from a given source s:

[Akrida, Gąsieniec, Mertzios, Spirakis, *TOCS*, 2017]

- first sort the time-labels non-decreasingly
- run a BFS-like search starting from s
- at every time-step t consider only edges currently available
- if you reach a new vertex at time t, keep its predecessor

An easy algorithm for computing all foremost paths from a given source s:

[Akrida, Gąsieniec, Mertzios, Spirakis, *TOCS*, 2017]

Algorithm 1 Foremost Temporal Paths from Source s

Let S be the array with the sorted time-labels
$R \leftarrow \{s\}$
for each $v \in V \setminus \{s\}$ **do**
    $pred[v] \leftarrow \emptyset$; $arr[v] \leftarrow \infty$ {Init.: Predecessor; Time Arrived}
for each time-label $t \in S$ **do**
    for each edge $e = (u, v)$ with $t \in \lambda(e)$ **do**
        if $u \in R$, $v \notin R$, and $arr[u] < t$ **then** {we reached v}
            $pred[v] \leftarrow u$; $arr[v] \leftarrow t$ {Predecessor; Time Arrived}
            $R \leftarrow R \cup \{v\}$

Metrics to optimize

An easy algorithm for computing all foremost paths from a given source s:

- easy adaptation of the static BFS algorithm
- running time $O(c(\lambda) \cdot \log(c(\lambda)))$
- due to the sorting of the labels

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Metrics to optimize

Polynomial algorithms exist also in the case of edges with traversal times for computing:

- shortest and foremost paths   [adaptations of Dijkstra's algorithm]
- fastest paths

[Bui-Xuan, Ferreira, Jarry, *Int. J. Found. Comp. Sci.*, 2003]

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Question: Are all "path-related" temporal problems tractable?

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- shortest and foremost paths [adaptations of Dijkstra's algorithm]
- fastest paths

[Bui-Xuan, Ferreira, Jarry, *Int. J. Found. Comp. Sci.*, 2003]

Question: Are all "path-related" temporal problems tractable?

Answer: Not all!

E.g. some temporal variations of:

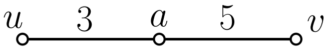
- connectivity problems
- reachability problems

Overview

- Temporal graphs
- Temporal paths
- Strongly connected components
- Temporal exploration
- Temporal design problems
- Stochastic temporal graphs
- Future research directions

Temporal strongly connected components

- We write $u \rightsquigarrow v$ if there exists a temporal path from u to v
- The relation $\rightsquigarrow$ is not symmetric: $u \rightsquigarrow v \;\not\Leftrightarrow\; v \rightsquigarrow u$



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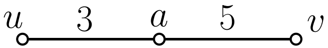
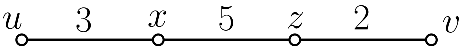


- and not transitive: $u \rightsquigarrow z,\ z \rightsquigarrow v \not\Rightarrow u \rightsquigarrow v$



$\Rightarrow$ the time dimension creates its own "level of direction"

Temporal strongly connected components

Recall:

Definition

A directed (static) graph G is strongly connected if there is a path in each direction between each pair of vertices of G.

Temporal strongly connected components

Recall:

Definition

A directed (static) graph G is strongly connected if there is a path in each direction between each pair of vertices of G.

A key property:

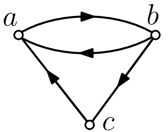
Observation

Let S be a (maximal) strongly connected subgraph and $u, v \in S$. If $P = (u, \dots, z, \dots, v)$ is a path from u to v then $z \in S$.

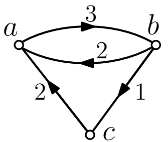
- Does this transfer to temporal graphs?

Temporal strongly connected components

static:



temporal:



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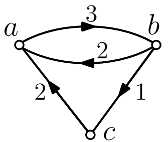

Temporal strongly connected components

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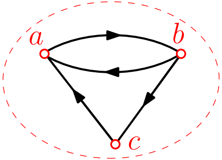
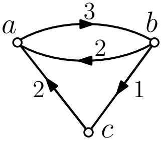
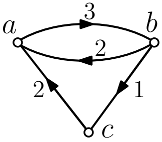


strongly
connected

temporal:



temporal:



Temporal strongly connected components

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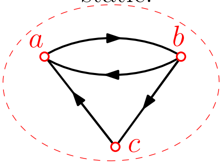

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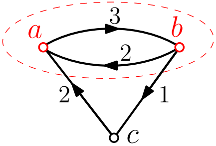
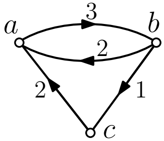
temporal:



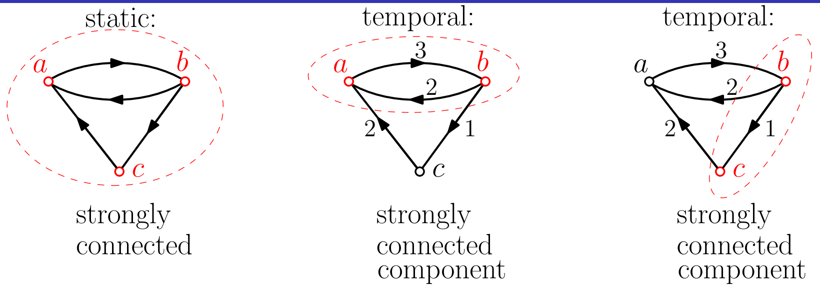
strongly connected component

temporal:



- $\{a, b\}$: direct temporal paths between a and b

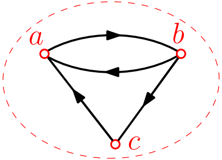
Temporal strongly connected components



- $\{a, b\}$: direct temporal paths between a and b
- $\{b, c\}$: the only temporal path from c to b passes through $a \notin \{b, c\}$

Temporal strongly connected components

static:



strongly
connected

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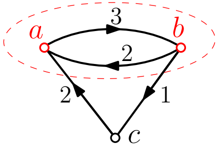
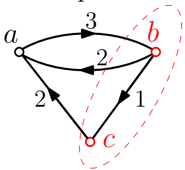


strongly
connected
component

temporal:



strongly
connected
component

- $\{a, b\}$: direct temporal paths between a and b
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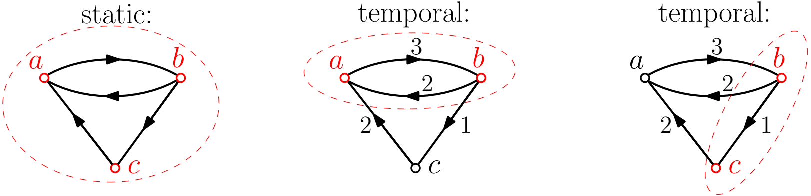
- $\{a, b, c\}$: no temporal path from a to c

Temporal strongly connected components

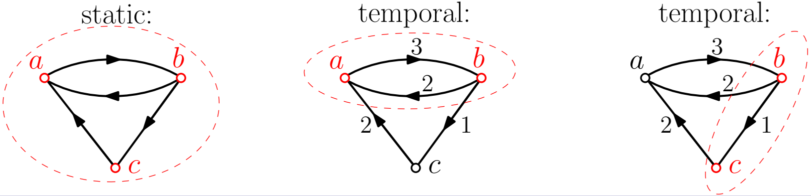


Definition (Bhadra, Ferreira, 2012)

An open strongly connected component (o-SCC) in a temporal graph is a set S of vertices such that $u \rightsquigarrow v$ for every $u, v \in S$.

Examples of an o-SCC: $\{a, b\}$, $\{b, c\}$

Temporal strongly connected components



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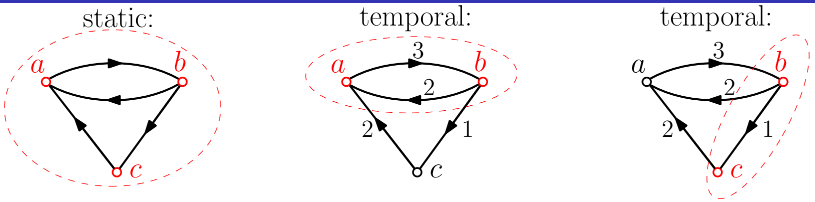
Definition (Bhadra, Ferreira, 2012)

A strongly connected component (SCC) in a temporal graph is a set S of vertices such that, for every $u, v \in S$, there is a temporal path from u to v that uses only vertices from S.

Example of a SCC: $\{a, b\}$

Temporal strongly connected components



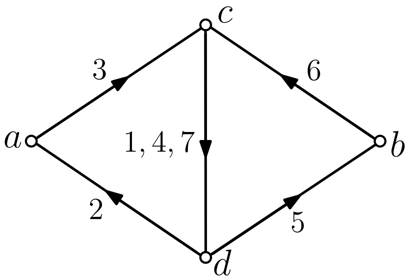
A difference to the static case:

- there can be a path between two vertices of the SCC (e.g. $\{a, b\}$) that traverses vertices outside the SCC (e.g. c)
- the same for an o-SCC (e.g. $\{b, c\}$)

Temporal strongly connected components

Further differences to the static case:

- two different SCCs can have common vertices



Temporal strongly connected components

Further differences to the static case:

- two different SCCs can have common vertices
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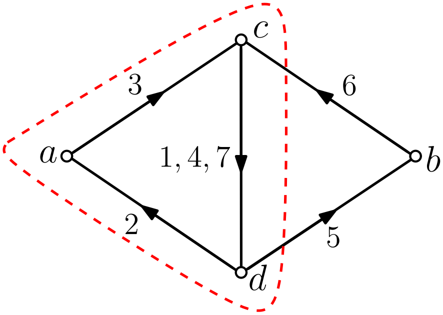

Temporal strongly connected components

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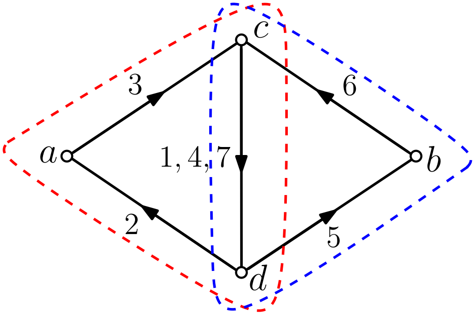

Temporal strongly connected components

Further differences to the static case:

- two different SCCs can have common vertices
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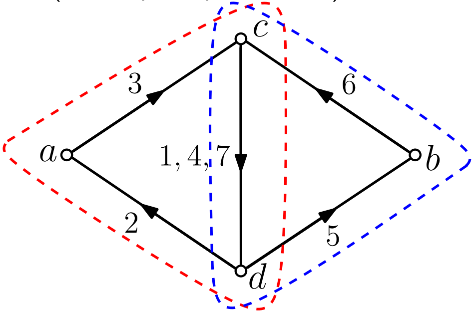
  - $\{a, b, c, d\}$ is not a SCC (no temporal path $b \rightsquigarrow a$)



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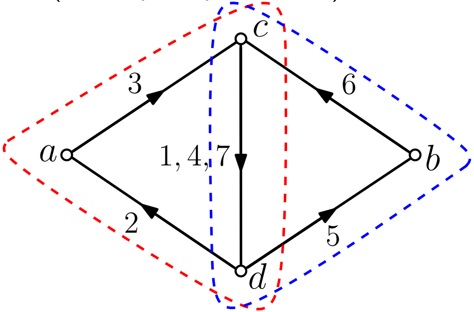

- Can we compute/verify temporal SCCs/o-SCCs efficiently?

Temporal strongly connected components

Theorem (Bhadra, Ferreira, 2012)

Given a vertex subset S of a temporal graph (G, λ), we can verify in polynomial time whether S is a SCC (resp. an o-SCC).

Temporal strongly connected components

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Given a vertex subset S of a temporal graph (G, λ), we can verify in polynomial time whether S is a SCC (resp. an o-SCC).

Proof.

- consider the induced (temporal) subgraph on S (resp. whole (G, λ))
- from every vertex $v \in S$ compute all foremost temporal paths
  - or all shortest / fastest paths, with any of the known algorithms
- if at least one vertex v does not reach the whole S:
  - then S is not a SCC (resp. an o-SCC)

□

Observation: similarly to static graphs

Temporal strongly connected components

Theorem

Given a temporal graph (G, λ), it is NP-hard to compute the maximum size of a SCC, even if all edges have one and the same label.

Theorem (Bhadra, Ferreira, 2012)

Given a temporal graph (G, λ), it is NP-hard to compute the maximum size of an o-SCC, even if all edges have two labels.

- Reduction from CLIQUE:
  Input: Graph G
  Goal: Find a clique of maximum size in G.

Overview

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- Stochastic temporal graphs
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Temporal exploration

Temporal Exploration Problem (TEXP) (Michail, Spirakis, 2014)

Input: *Temporal graph* (G, λ) *and source vertex s*

Goal: *Visit each vertex at least once with a temporal walk that minimizes the arrival time (possibly revisiting vertices)*

Its "static analogue": Graphic Traveling Salesman Problem

- $\frac{13}{9}$-approximation algorithm [Mucha, *Th. Comp. Syst.*, 2014]

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Observation

The decision version in the static case can be solved in linear time.

Proof.

- A static graph G is explorable $\Leftrightarrow$ G is connected.

$\Rightarrow$ Check connectivity in G by DFS. $\square$

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Observation

If a temporal graph (G, λ) *is connected at every time* t*, then it is always explorable.*

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Observation

If a temporal graph (G, λ) is connected at every time t, then it is always explorable.

However:

Theorem (Michail, Spirakis, *MFCS*, 2014)

The decision version in the temporal case is NP-complete.

Temporal exploration

Theorem (Erlebach, Hoffmann, Kammer, *ICALP*, 2015)

There exists an infinite family of temporal graphs that require $\Omega(n^2)$ time to be explored.

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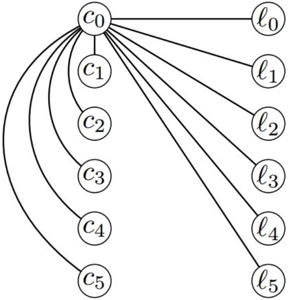
Proof.

- Let $V = \{c_j, \ell_j : 0 \leq j \leq n-1\}$ be the vertex set of G
- The "snapshot" of G at time $t \geq 0$ is a star with center $c_{t \bmod n}$



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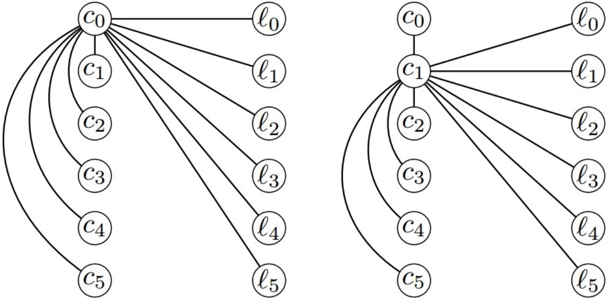

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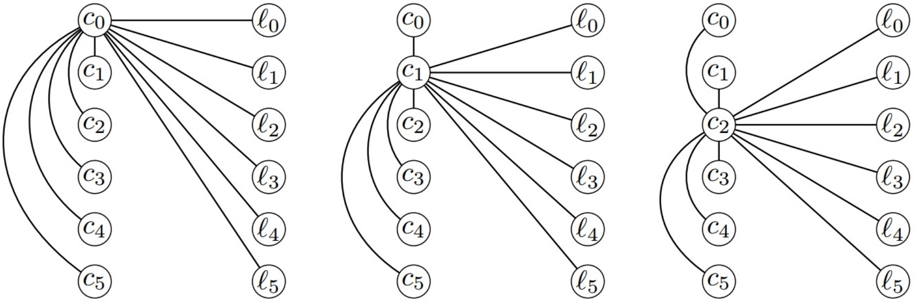
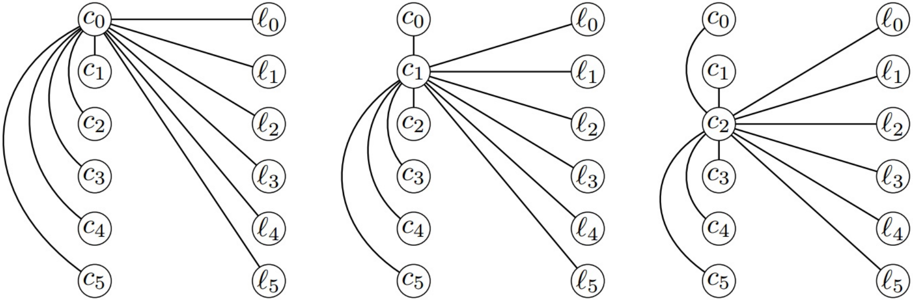


Proof (continued).



- If the exploring agent is at a vertex that is not the current center:
  - it can only wait or travel to the current center
- If it moves, at the next step it will be again not in the current center

$\Rightarrow$ to go from ℓ_i to ℓ_j, $i \neq j$, n steps are needed:

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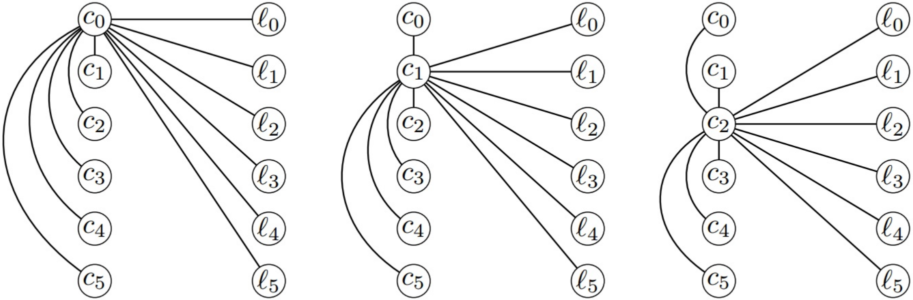
Proof (continued).



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$\Rightarrow$ the total number of steps is $\Omega(n^2)$

Temporal exploration

Modifying the reduction used to prove NP-completeness, the result can be strongly amplified:

Theorem (Erlebach, Hoffmann, Kammer, *ICALP*, 2015)

Approximating TEXP with ratio $O(n^{1-\varepsilon})$ is NP-hard.

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The previous reduction can be also "parameterized":

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For every $\Delta > 0$, there exists an infinite family of temporal graphs with maximum degree Δ that require $\Omega(\Delta n)$ time to be explored.

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For every $\Delta > 0$, there exists an infinite family of temporal graphs with maximum degree Δ that require $\Omega(\Delta n)$ time to be explored.

Furthermore, on restricted classes of underlying graphs:

Theorem (Erlebach, Hoffmann, Kammer, *ICALP*, 2015)

Any temporal graph whose underlying graph has treewidth at most k, can be explored in $O(n^{1.5}k^2 \log n)$ time.

- Restricting the problem on the special class of underlying star graphs.
- Motivation: inspired by the well-known Traveling Salesperson Problem (TSP): "Given a list of cities and the distances between each pair of cities, what is the shortest possible route that visits each city and returns to the origin?", i.e. find a min-cost Hamiltonial cycle.
  - What if the salesperson has particular temporal constraints, e.g. (s)he can only go from city A to city B on Mondays or Tuesdays?
  - What if (s)he needs to return to their home town after visiting each city?
  - Can the salesperson decide whether (s)he can visit all towns and finally return to their home town by a certain day or time?

Definition (Temporal Star)

A temporal star is a temporal graph (G_s, λ) on a star graph $G_s = (V, E)$.

Definition (Exploration of a temporal star)

A (partial) exploration of a temporal star is a journey J that starts and ends at the center of G_s which visits some nodes of G_s; its size $|J|$ is the number of nodes of G_s that are visited by J.

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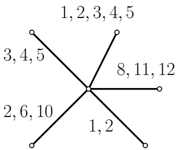
- We "enter" (resp. "exit") an edge when we cross it from center to leaf (resp. leaf to center) at a time on which the edge is available.
- We can assume that in an exploration the entry to any edge e is followed by the exit from e at the earliest possible time. Waiting at a leaf (instead of exiting as soon as possible) does not help in exploring more edges.



Star Exploration Problem ($\textsc{StarExp}(k)$) (Akrida, Mertzios, Spirakis, *CIAC*, 2019)

Input: *A temporal star* (G_s, λ) *such that every edge has at most k labels.*

Question: *Is* (G_s, λ) *explorable?*

Maximum Star Exploration Problem ($\textsc{MaxStarExp}(k)$) (Akrida et al., *CIAC*, 2019)

Input: *A temporal star* (G_s, λ) *such that every edge has at most k labels.*

Output: *A (partial) exploration of* (G_s, λ) *of maximum size.*

- $\textsc{MaxStarExp}(3)$ can be efficiently solved in $O(n \log n)$ time
- $\textsc{StarExp}(k)$ is NP-complete, when $k \geq 6$
- $\textsc{MaxStarExp}(k)$ is APX-complete, when $k \geq 4$

	Maximum number of labels per edge					
	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k \geq 6$
$\textsc{StarExp}(k)$	No	$O(n \log n)$	$O(n \log n)$	?	?	NP-c
$\textsc{MaxStarExp}(k)$	No	$O(n \log n)$	$O(n \log n)$	APX-c	APX-c	APX-c

Theorem (Akrida, Mertzios, Spirakis, *CIAC*, 2019)

$\textsc{MaxStarExp}(3)$ *can be efficiently solved in* $O(n \log n)$ *time*

Proof (sketch).

- problem is reducible to the Interval Scheduling Maximization Problem (ISMP):
  Input: A set of intervals, each with a start and a finish time.
  Output: Find a max-size set of non-overlapping intervals.

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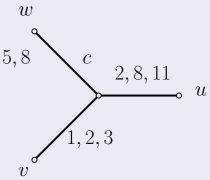
  - Any edge with a single label can be ignored.



Interval 1: [2,8]
Interval 2: [8,11]
Interval 3: [1,2]
Interval 4: [2,3]
Interval 5: [5,8]

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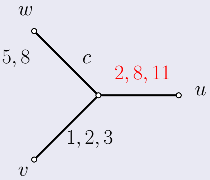

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  Input: A set of intervals, each with a start and a finish time.
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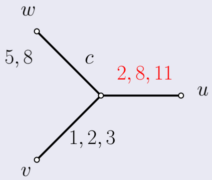
- Any (partial) exploration of (G_s, λ) corresponds to a set of non-overlapping intervals of the same size as the exploration, and vice versa.



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Temporal exploration

The travelling salesperson who returns home

Theorem (Akrida, Mertzios, Spirakis, *CIAC*, 2019)

$\textsc{MaxStarExp}(3)$ *can be efficiently solved in* $O(n \log n)$ *time*

Proof (sketch, continued).

Greedy optimal solution for ISMP:

1. Start with the set $\mathcal{I}$ of all intervals $(|\mathcal{I}| \leq 2(n-1))$. Select the iterval, I, with the earliest finish time.
2. Remove from $\mathcal{I}$ the interval I and all overlapping intervals.
3. Repeat until $\mathcal{I}$ is empty.

Time needed: $(|\mathcal{I}| \log |\mathcal{I}|) = O(n \log n)$ □

Temporal exploration: further work

- Minimum cost exploration; complete directed temporal graph with edge weights from $\{1, 2\}$
  [Michail and Spirakis. Traveling salesman problems in temporal graphs, *TCS*, 2016.]
- Exploration of constantly connected dynamic graphs; underlying cactus graph
  [Ilcinkas,Klasing, Wade. Exploration of constantly connected dynamic graphs based on cactuses, *SIROCCO*, 2014.]
- Exploration of temporal graphs of small pathwidth; NP-completeness
  [Bodlaender and van der Zanden. On exploring always-connected temporal graphs of small pathwidth, *Information Processing Letters*, 2014.]
- Exploration of temporal graphs using temporal paths with non-strictly increasing labels
  [Erlebach, Spooner. Non-strict Temporal Exploration, *SIROCCO*, 2020.]

Overview

- Temporal graphs
- Temporal paths
- Strongly connected components
- Temporal exploration
- Temporal design problems
- Stochastic temporal graphs
- Future research directions

Temporal design problems

So far:

- we were given the input temporal graph (G, λ) and
- we were asked to optimize some metric (e.g. a foremost path)

Many times the problem is different:

- we are given a graph G and
- we are asked to construct a time-labeling λ such that:
  - λ minimizes some cost function and
  - (G, λ) satisfies some connectivity constraints

[Akrida, Gąsieniec, Mertzios, Spirakis, *TOCS*, 2017]

[Mertzios, Michail, Spirakis, *Algorithmica*, 2019]

Temporal design problems

In many scheduling problems:

- the provided graph topology G represents a given static specification
  - e.g. available bus routes in the city center
- the aim is to organize a temporal schedule on this specification, e.g.
  - when the buses should be in which stop
  - such that every pair of stops is connected via a route
- while minimizing some cost function
  - e.g. with as few buses as possible

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  - such that every pair of stops is connected via a route
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Creating and maintaining a connection does not come for free, e.g.:

- edge "rentals" / toll roads
- in wireless sensor networks the connection cost depends on the power consumption of the vertices awake

Temporal design problems

We mainly study the following cost functions of a time-labeling λ:

1. temporal cost κ: the total number of labels on all edges
   - a centralized measure of cost
2. temporality τ: the maximum number of labels per edge
   - a distributed / decentralized measure of cost in the temporal network
3. as well as trade-offs between the age $\alpha(\lambda)$ and these parameters

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and two fundamental connectivity properties:

1. preserve in (G, λ) all reachabilities in G
   - if v is reachable from u in $G \Rightarrow u \rightsquigarrow v$ in (G, λ)
2. preserve in (G, λ) all paths in G
   - G has a path $P \Rightarrow (G, \lambda)$ has a temporal path on the same edges as P

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Notation (combining cost function & connectivity property):

- $\kappa(G, \mathit{reach})$, $\tau(G, \mathit{all\ paths})$, $\tau(G, \mathit{all\ paths}, \alpha(\lambda))$, etc.

Temporal cost for preserving all reachabilities

Cost function: total number κ of labels

Theorem (Mertzios, Michail, Spirakis, *Algorithmica*, 2019)

Let $d(G)$ denote the (static) diameter of the directed graph G. The problem of computing $\kappa(G, reach, d(G))$ is APX-hard, even when each directed cycle of G has length at most 2.

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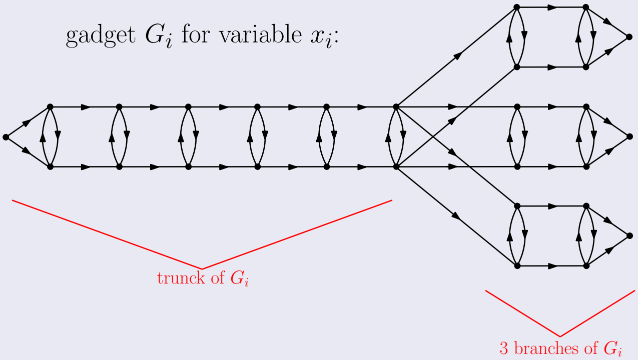
Proof (sketch).

- reduction from Max-XOR(3):
  - formula ϕ with n variables and m clauses
  - XOR-clauses $(\ell_i \oplus \ell_j)$ with two literals each: $(\ell_i \oplus \ell_j) = 1 \;\Leftrightarrow\; \ell_i \neq \ell_j$
  - each variable appears in at most 3 clauses $\;\Rightarrow\;$ $m \leq \frac{3}{2}n$
  - the goal is to find a truth assignment τ with the maximum number $|\tau(\phi)|$ of XOR-satisfied clauses
- from ϕ we construct a graph G_ϕ and we prove:
  - $|\tau(\phi)| \geq k \;\Leftrightarrow\; \kappa(G_\phi, reach, d(G_\phi)) \leq 39n - 4m - 2k$

Temporal cost for preserving all reachabilities

Proof (sketch, continued).

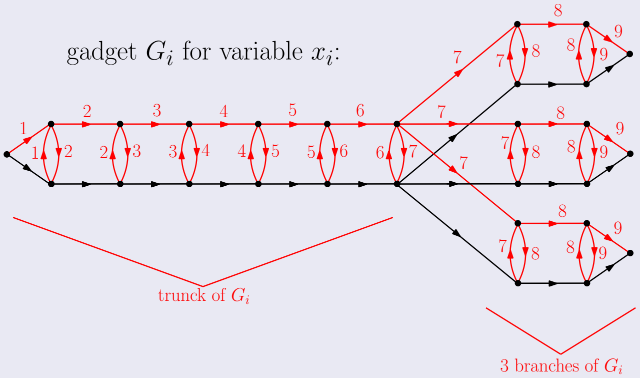
- diameter $d(G_i) = 9$



Temporal cost for preserving all reachabilities

Proof (sketch, continued).

- diameter $d(G_i) = 9$
- to achieve a maximum label 9 we have two choices:
  - $x_i = 0$



Proof (sketch, continued).

- diameter $d(G_i) = 9$
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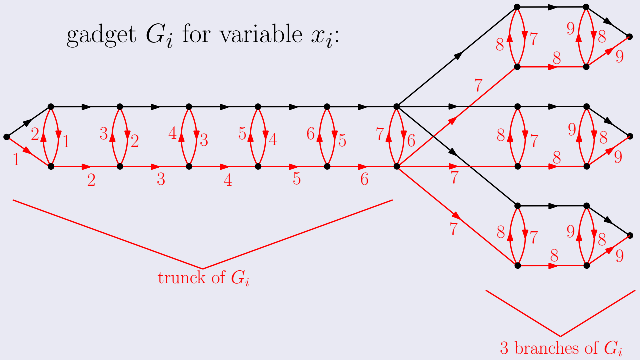
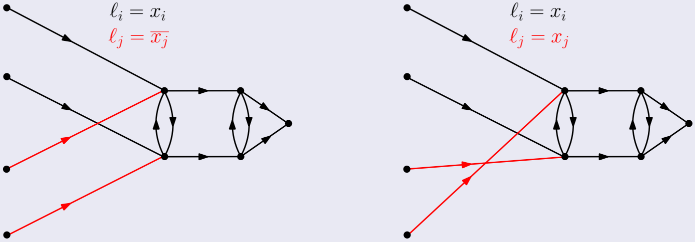


Temporal cost for preserving all reachabilities

Proof (sketch, continued).

- for every clause $(\ell_i \oplus \ell_j)$ where:
  - ℓ_i corresponds to the pth appearance of x_i $\quad (p \in \{1,2,3\})$
  - ℓ_j corresponds to the qth appearance of x_j $\quad (q \in \{1,2,3\})$

  we identify the pth branch of G_i and the qth branch of G_j as follows:



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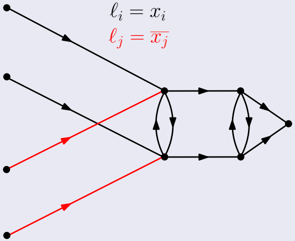

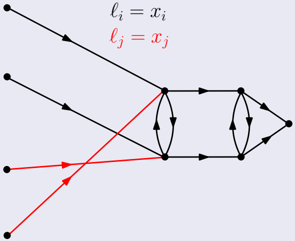

- $\ell_i \neq \ell_j \;\Leftrightarrow\;$ the correct "tracks" of these branches are labeled
- otherwise we use both "tracks" $\;\Rightarrow\;$ pay more labels

$\square$

Temporal cost for preserving all reachabilities

A simple approximation algorithm:

- the reachability number of $u \in V$:

$$r(u) = |\{v \in V : v \text{ is reachable from } u\}|$$

- the total reachability number: $r(G) = \sum_{u \in V} r(u)$

Theorem (Mertzios, Michail, Spirakis, *Algorithmica*, 2019)

A $\frac{r(G)}{n-1}$-approximation for $\kappa(G, reach, d(G))$ can be computed in polynomial time for connected graphs G.

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Proof.

- Compute from every $u \in V$ a temporal out-tree

$\Rightarrow$ all reachabilities are maintained with $\leq r(G)$ labels

- OPT$\geq n - 1 \;\Rightarrow$ approximation ratio $\frac{r(G)}{n-1}$ $\square$

Nearly cost-optimal design for preserving all reachabilities

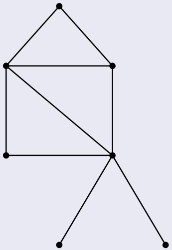
Theorem (Akrida, Gąsieniec, Mertzios, Spirakis, *TOCS*, 2017)

Given a connected undirected graph $G = (V, E)$ of $n \geq 2$ vertices, we can construct a labelling λ of cost $c(\lambda) = 2n - 3$ that preserves all reachabilities on G in polynomial time.

Proof.

- We consider a fixed, arbitrary spanning tree T of G and let a leaf node w be its root.

$G = (V, E)$



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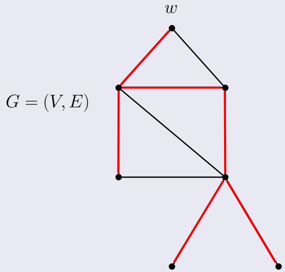

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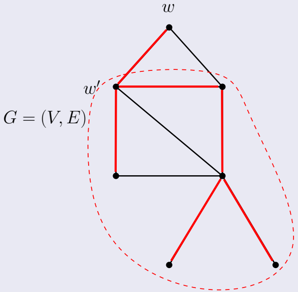

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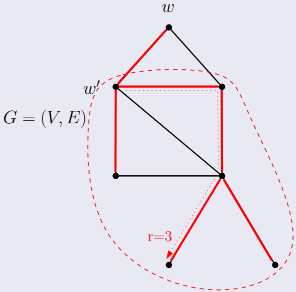
- Let r length of longest path from w' to any leaf of T'.



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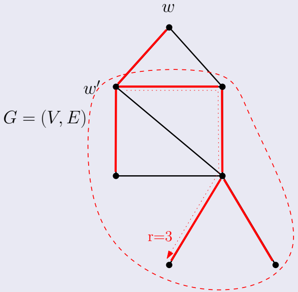
- Let w' be the single child of w in T and let T' be the subtree of T that is rooted at w'.
- Let r length of longest path from w' to any leaf of T'.
- We assign labels to the edges of T as follows.



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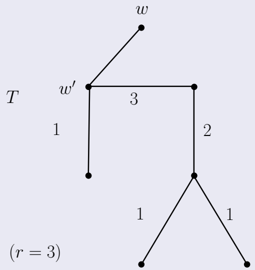
Proof.

Going upwards.
Any edge of T' incident to a leaf gets label 1. Any edge $e = \{u, v\}$ of T', with $d(w', v) = d(w', u) + 1$, where the subtree T^* rooted at v has been labelled going upwards towards w', gets a label $l_e = max\{\text{all labels in } T^*\} + 1$.



$\square$

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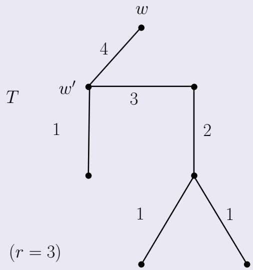
Proof.

The edge $\{w, w'\}$.
We label the edge $\{w, w'\}$ of T with the single label $r + 1$.



$\square$

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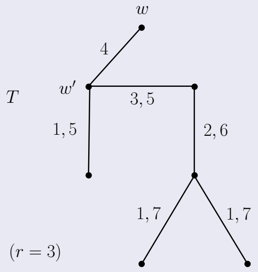
Proof.

Going downwards.

Any edge of T' incident to w' gets a label $r + 2$. Any edge e of T' in a path from w' to a leaf of T', the *parent edge* of which has been labelled, going downwards, with label l', gets a label $l_e = l' + 1$.



$\square$

Removal cost for preserving all reachabilities

The "inverse" design problem:

- given a temporal graph (G, λ) that maintains all reachabilities of G
- remove the maximum number of labels by maintaining reachabilities
- removal cost $r(G, \lambda)$

Removal cost for preserving all reachabilities

Theorem (Akrida, Gąsieniec, Mertzios, Spirakis, *TOCS*, 2017)

The problem of computing $r(G, \lambda)$ is APX-hard on undirected graphs G.

Removal cost for preserving all reachabilities

Theorem (Akrida, Gąsieniec, Mertzios, Spirakis, *TOCS*, 2017)

The problem of computing $r(G, \lambda)$ is APX-hard on undirected graphs G.

Proof (sketch).

- reduction from monotone Max-XOR(3):
  - same as Max-XOR(3) but no variable is negated
- from ϕ we construct a graph G_ϕ and we prove:
  - $|\tau(\phi)| \geq k \;\Leftrightarrow\; r(G_\phi, \lambda) \geq 9n + k$
  - assuming a PTAS for computing $r(G_\phi, \lambda)$, we obtain a PTAS for monotone Max-XOR(3)
  - Contradiction; monotone Max-XOR(3) is APX-hard [Alimonti and Kann, *CIAC*, 1997]

Proof (sketch, continued).

Construction of G_ϕ, λ

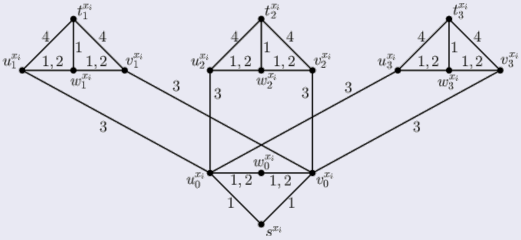


Figure: The gadget $G_{\phi,i}$ for the variable x_i.

- For every $p \in \{1, 2, 3\}$ we associate the pth appearance of the variable x_i in a clause of ϕ with the pth branch of $G_{\phi,i}$.

Removal cost for preserving all reachabilities

Proof (sketch, continued).

Construction of G_ϕ, λ

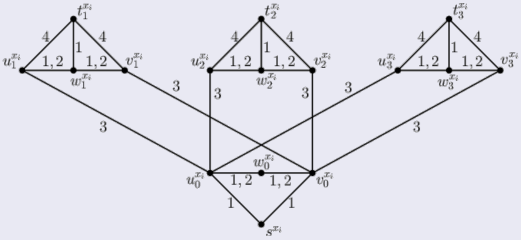


Figure: The gadget $G_{\phi,i}$ for the variable x_i.

- For every pair of vertices $w_p^{x_i}, w_q^{x_j}$, $p, q \in \{0, 1, 2, 3\}$, $i, j \in \{1, 2, \ldots, n\}$ add an edge $e = \{w_p^{x_i}, w_q^{x_j}\}$ with label 7.

Removal cost for preserving all reachabilities

Proof (sketch, continued).

Construction of G_ϕ, λ

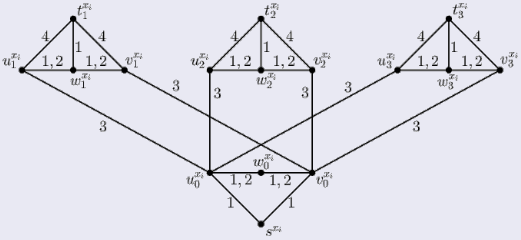


Figure: The gadget $G_{\phi,i}$ for the variable x_i.

- For every pair of vertices $t_p^{x_i}, t_q^{x_j}$, $p, q \in \{1, 2, 3\}$, $i, j \in \{1, 2, \ldots, n\}$ add an edge with label 7.

Removal cost for preserving all reachabilities

Proof (sketch, continued).

Construction of G_ϕ, λ

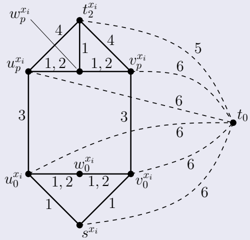


Figure: The addition of vertex t_0. There exists in G_ϕ also the edge $\{t_0, w_0^{x_n}\}$ with label 5.

- Add vertex t_0 adjacent to all vertices $\{s^{x_i}, t_1^{x_i}, t_2^{x_i}, t_3^{x_i}, u_p^{x_i}, v_p^{x_i} : 1 \leq i \leq n,\ 0 \leq p \leq 3\}$ with label 5 or 6.

Removal cost for preserving all reachabilities

Proof (sketch, continued).

Construction of G_ϕ, λ

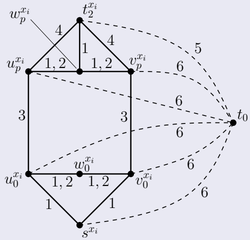


Figure: The addition of vertex t_0. There exists in G_ϕ also the edge $\{t_0, w_0^{x_n}\}$ with label 5.

- Add the edge $t_0, w_0^{x_n}$ with label 5.

Removal cost for preserving all reachabilities

Proof (sketch, continued).

Construction of G_ϕ, λ

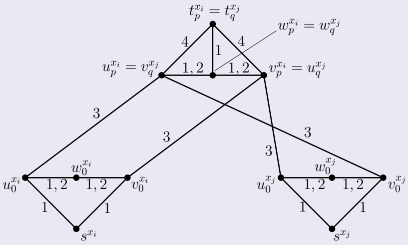


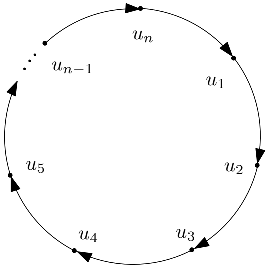
Figure: The gadget for the clause $(x_i \oplus x_j)$.

- Consider now a clause $\alpha = (x_i \oplus x_j)$ of ϕ. Assume that the variable x_i (resp. x_j) of α corresponds to the pth (resp. to the qth) appearance of x_i (resp. of x_j) in ϕ. Then we identify the vertices $u_p^{x_i}, v_p^{x_i}, w_p^{x_i}, t_p^{x_i}$ of the pth branch of $G_{\phi,i}$ with the vertices $v_q^{x_i}, u_q^{x_i}, w_q^{x_i}, t_q^{x_i}$ of the qth branch of $G_{\phi,j}$, respectively.

□

Temporality of the ring C_n

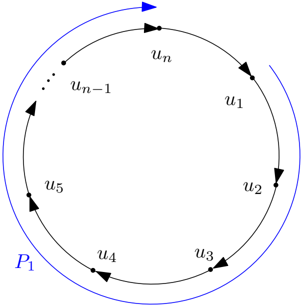
A very different cost function: maximum number τ of labels per edge



Temporality of the ring C_n

A very different cost function: maximum number τ of labels per edge

- increasing labels on P_1 $\Rightarrow$ decreasing labels from (u_{n-1}, u_n) to (u_1, u_2)



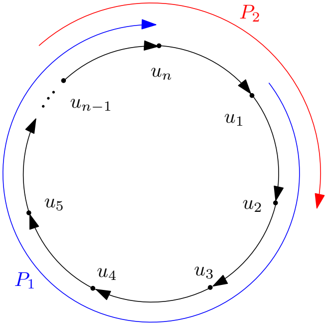
Temporality of the ring C_n

A very different cost function: maximum number τ of labels per edge

- increasing labels on P_1 $\Rightarrow$ decreasing labels from (u_{n-1}, u_n) to (u_1, u_2)
- P_2 uses first (u_{n-1}, u_n), then (u_1, u_2)

$\Rightarrow$ increasing pair of labels on these edges

- To preserve both P_1, P_2 we need 2 labels on at least one of these two edges $\Rightarrow$ $\tau(C_n, \textit{all paths}) \geq 2$



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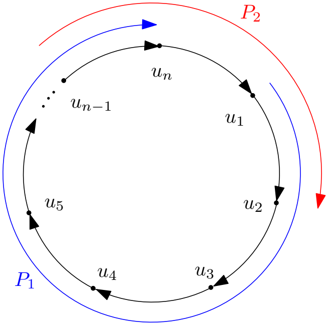


- The labeling that assigns to each edge (u_i, u_{i+1}) the labels $\{i, n+i\}$ preserves all simple paths, i.e. $\tau(C_n, \textit{all paths}) \leq 2$

$\Rightarrow$ $\tau(C_n, \textit{all paths}) = 2$

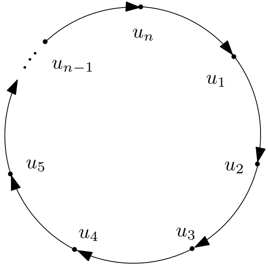
- The maximum label is $2n$ (can be "tuned" to $2n-2$)

Temporality of the ring C_n

Restricting the age

What if we restrict the age to $\alpha(\lambda) = n - 1$?

- Assume that some edge e of C_n misses label $i \in \{1, 2, \ldots, n-1\}$
- Then there exists a temporal path on C_n that needs label i on edge e to finish by time $n - 1$



Temporality of the ring C_n

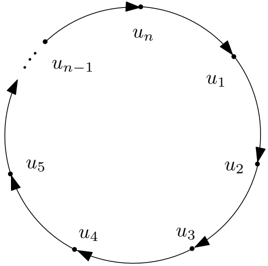
Restricting the age

What if we restrict the age to $\alpha(\lambda) = n-1$?

- Assume that some edge e of C_n misses label $i \in \{1, 2, \ldots, n-1\}$
- Then there exists a temporal path on C_n that needs label i on edge e to finish by time $n-1$

$\Rightarrow$ the optimal labeling assigns $\{1, 2, , \ldots, n-1\}$ to all edges of C_n

$\Rightarrow$ $\tau(C_n, \textit{all paths}, n-1) = n-1$



More generally:

Theorem (Mertzios, Michail, Spirakis, *Algorithmica*, 2019)

If G is a directed ring C_n and $\alpha(\lambda) = (n-1) + k$, where $1 \leq k \leq n-1$, then

$$\tau(G, \textit{all paths}, \alpha) = \Theta(n/k).$$

In particular: $\lfloor \frac{n-1}{k+1} \rfloor + 1 \leq \tau(G, \textit{all paths}, \alpha) \leq \lceil \frac{n}{k+1} \rceil + 1.$

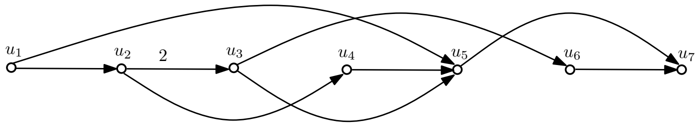
Temporality of a DAG

A topological sort of a digraph G:

- a linear ordering of its vertices, where
- if G contains an arc (u, v) then u appears before v

It is known:

- a digraph G can be topologically sorted $\Leftrightarrow$ G is a DAG



Temporality of a DAG

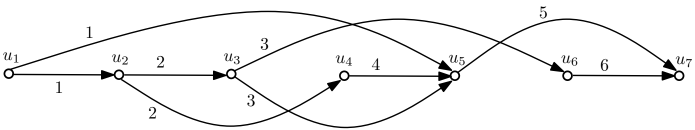
Lemma (Mertzios, Michail, Spirakis, *Algorithmica*, 2019)

If G is a DAG then $\tau(G, \text{all paths}) = 1$.

Proof.

- Take a topological sort $u_1, u_2, \ldots, u_n$ of G
- Give label i to every edge (u_i, u_j), where $i < j$.

$\square$



Temporality: preserving all reachabilities

Theorem (Mertzios, Michail, Spirakis, *Algorithmica*, 2019)

Let G be an undirected (or strongly connected directed) graph. Then $\tau(G, reach) \leq 2$.

Proof.

- pick an arbitrary vertex v
- let v have (static) distance at most k to all other vertices
- build a temporal in-tree to vertex v with labels $\{1, 2, \dots, k\}$
- from v build a temporal out-tree to vertex v with labels $\{k+1, k+2, \dots, 2k\}$

$\Rightarrow$ all vertices remain temporally connected with 2 labels per edge

$\square$

Temporality: preserving all reachabilities

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Similarly to the analysis for DAGs:

Theorem (Mertzios, Michail, Spirakis, *Algorithmica*, 2019)

Let G be a directed graph. Then $\tau(G, reach) = \max_{C \in \mathcal{C}(G)} \tau(C, reach)$, where $\mathcal{C}(G)$ is the set of strongly connected components of G.

Temporality: preserving all reachabilities

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Therefore:

Corollary

$\tau(G, reach) \leq 2$ for every directed or undirected graph.

That is: we can preserve all reachabilities with at most 2 labels per edge.

Overview

- Temporal graphs
- Temporal paths
- Strongly connected components
- Temporal design problems
- Temporal exploration
- Stochastic temporal graphs
- Future research directions

Stochastic temporal graphs

Levels of knowledge about the network evolution:

- whole temporal graph given in advance
- adversary who reveals it snapshot-by-snapshot at every time step
- intermediate knowledge setting, captured by **stochastic temporal graphs**, where the network evolution is given by a probability distribution that governs the appearance of each edge over time

Stochastic temporal graphs

Models of randomness:

- "Selection from a pool of labels": the labels assigned to the edges of the underlying graph are chosen independently and uniformly at random from a set of available time labels.
  - ephemeral random dynamic networks
- "Fixed probability": each edge exists at each time-step with a certain probability
  - opportunistic mobile networks
- "Memory effect": appearance probability of a particular edge at a given time-step t depends on the appearance (or absence) of the same edge at the previous $k \geq 1$ time steps
  - faulty network communication

- Upper and lower bounds on the Temporal Diameter, i.e., expected maximum temporal distance, of the complete graph with a single label per edge
  - Note that the complete graph is the only graph for which *preserving all reachabilities* is guaranteed with random labels, even with 1 label per edge.

  [Akrida, Gąsieniec, Mertzios, Spirakis *JPDC*, 2016]
- Bound on the smallest number of random labels per edge that guarantees preservation of reachability with high probability ⇒ upper bound on a measure of how "good" the best random assignment is compared to the best deterministic one.
  [Akrida, Gąsieniec, Mertzios, Spirakis *JPDC*, 2016]
- High removal profit with high probability for complete graphs and random graphs.
  [Akrida, Gąsieniec, Mertzios, Spirakis *TOCS*, 2017]
- Flows in uniform random temporal networks: characterisation of networks that block any flow from arriving to the target.
  [Akrida, Czyzowicz, Gąsieniec, Kuszner, Spirakis, *J. Comp. Syst. Sci.*, 2019]

Stochastic temporal graphs
Fixed probability per time step

- Model of opportunistic mobile networks as a type of random temporal networks, where each edge exists at each time-step with a fixed probability; proof of existence of small diameter.
  [Chaintreau, Mtibaa, Massoulié, Diot, *CoNEXT*, 2007]
- Dijkstra-like polynomial-time algorithm for computing the arrival time of a best policy for choosing an source-target journey
  [Basu, Bar-Noy, Ramanathan, Johnson, *arXiv*, 2010]
- Cost-effective data dissemination approach in disruption tolerant networks, based on a centrality metric.
  [Gao, Cao, *INFOCOM*, 2011]
- Efficient content dissemination in opportunistic social networks broken down into 'temporal communities', i.e., groups of people meeting periodically.
  [Pietiläinen, Diot, *MOBIHOC*, 2012]

Stochastic temporal graphs

Probability per time step with dependence of history

- Case of each edge existing at each time-step with probability dependent on the previous time-step: upper bounds for the flooding time and tight characterizations of the graphs on which the flooding time is constant
  [Clementi, Macci, Monti, Pasquale, Silvestri, *SIDMA*, 2010]

- Investigation of the complexity of two temporal path problems, namely computing the expected arrival time of a foremost source-target journey and of a best policy for choosing a particular source-target journey.
  [Akrida, Mertzios, Nikoletseas, Raptopoulos, Spirakis, Zamaraev, *J. Comp. Syst. Sci.*, 2020]

- Temporal graphs
- Temporal paths
- Strongly connected components
- Temporal design problems
- Temporal exploration
- Stochastic temporal graphs
- Future research directions

Research Directions

- Find constant-factor approximations for the various temporal graph design problems
- Other natural connectivity properties subject to which optimization is to be performed
- Efficient deterministic/randomized/approximation algorithms on special temporal graph classes, i.e. by restricting:
  - the underlying topology G and/or
  - the temporal pattern with which the time-labels appear
    (a new dimension with no previous static analogue!)

Research Directions

- Natural non-path temporal problems
  - the notion of a "Δ-temporal clique" in social networks:
    "a set of nodes and a time interval such that all pairs interact at least every Δ during this interval"
    [Viard, Latapy, Magnien, *ASONAM*, 2015]

Research Directions

- Natural non-path temporal problems
  - the notion of "temporal matchings" has been studied before:
    "a collection of edges and time steps of their existence such that the edges are a matching and each edge appears at a different time-step"
    [Michail, Spirakis, *TCS*, 2016]

Research Directions

- Natural non-path temporal problems
  - more recently defined temporal analogues of "vertex cover":
    "a collection of vertex appearances that cover every edge at least once" and "a collection of vertex appearances and a time length so that every edge is covered at least once within every time interval of the given length"
    [Akrida, Mertzios, Spirakis, Zamaraev, *J. Comp. Syst. Sci.*, 2020]

Research Directions

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- Our results so far are a first step towards answering this fundamental question:

  To what extent can algorithmic and structural results
  of graph theory be carried over to temporal graphs?

Thank you for your attention!