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# Ranking Bracelets in Polynomial time

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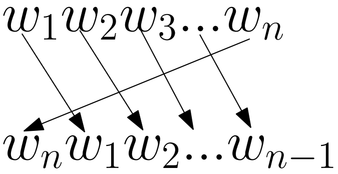
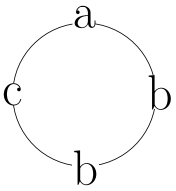
# What are Necklaces?

- A **necklace** is an equivalence class of words under the **cyclic shift** operation.
- The canonical representative of a necklace is the **lexicographically smallest word** in the equivalence class.



$w_1w_2w_3\ldots w_n$

$w_nw_1w_2\ldots w_{n-1}$



**abbc**
bbca
bcab
cabb

# What are Bracelets?

- A **bracelet** is an equivalence class of words under the **cyclic shift** and **reflection** operations.
- The canonical representation of a bracelet is the **lexicographically smallest word** in the equivalence class.

| unreflected | reflected |
|---|---|
| **abbc** | cbba |
| bbca | *acbb* |
| bcab | bacb |
| cabb | bbac |

# What is ranking?

- The ranking problem asks, given an object $o$ in a strictly ordered set $S$, how many members of S are smaller than $o$.
- For the set of bracelets of a given length $n$ over an alphabet of size $k$ ($\mathrm{B}(n, k)$), the ordering is defined over the canonical representations.

1. aaaa
2. aaab
3. aaac
4. aabb
5. aabc
6. aacc
7. abab
8. abac
9. abbb
**10. abbc**
11. abcb
12. abcc
13. acac
14. acbc
15. acccc
16. bbbb
17. bbbc
18. bbcc
19. bcbc
20. bccc
21. cccc

- Note that there are approximately $O(k^n)$ bracelets, making explicit generation of the set unfeasible for large values of $n$ and $k$.

# What is Unranking?

- The unranking problem can be thought of as the inverse of the ranking problem.
- The unranking problem asks, for a strictly ordered set $S$ what is the element at the $i^{th}$ position.
- For bracelets (and other classes of cyclic words) the exponential size of these sets makes the naive approach impractical.

# Ranking Cyclic Words

- The problem of ranking classes of cyclic words originates from the problem of ranking de Bruijn Sequences [2].
- The first class of cyclic words to be ranked was **Lyndon words** (aperiodic necklaces).
- This was generalised to ranking necklaces [3, 4] in quadratic time.
- More recently, an algorithm to rank the class of **Fixed density** necklaces in cubic time has been presented [1].

| Class | Solved by | Best Run time |
|---|---|---|
| Lyndon words | Kociumaka et. al. [2] | $O(n^2)$ ([4]) |
| Necklaces | Kopparty et. al. [3] | $O(n^2)$ ([4]) |
| Fixed Density Necklaces | Hartman and Sawada | $O(n^3)$ ([1]) |

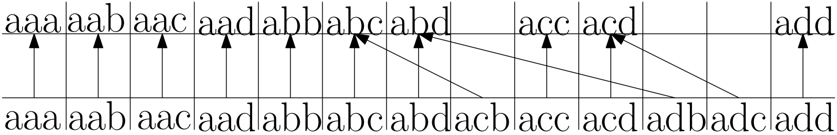
Table 1: Unranking algorithms for all sets have been implemented with an additional factor of $O(n \log k)$ [1, 4]

# Our Results

- We provide an $O(n^4 k^2)$ time algorithm to rank a word $w$ among the set of all bracelets of length $n$ over the alphabet of size $k$.
- Further, we provide an $O(n^5 \cdot k^2 \cdot \log(k))$ time algorithm to unrank bracelets.

# Bracelets and Necklaces

- Observe that **every** bracelets corresponds to either one or two necklace classes.
- A bracelet is **Palindromic** if it only corresponds to a single necklace class.
- A bracelet is **Apalindromic** if it corresponds to two necklace classes.
- An apalindromic bracelet $b = n_1 \cup n_2$ **encloses** a word $w$ if $n_1 < w < n_2$.



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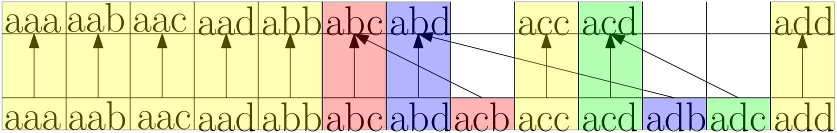

aaa aab aac aad abb abc abd acc acd add

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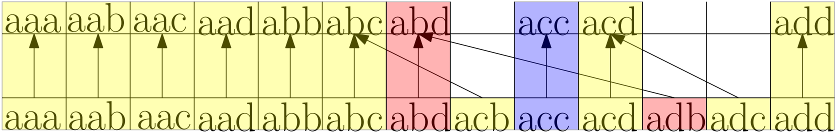

# High Level Idea

- The number of bracelets smaller than $w$ can be split into three categories:
  - The number of palindromic bracelets smaller than $w$ ($RP(w)$).
  - The number of apalindromic bracelets smaller than $w$ that do not enclose $w$ ($RA(w)$).
  - The number of bracelets enclosing $w$ ($RE(w)$).

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  - The number of bracelets enclosing $w$ ($RE(w)$).
- **Idea:** count each of these sets separately and add them together.

# Counting the number of apalindromic bracelets

- Rather than computing the number of apalindromic bracelets that do not enclose $w$ directly, we can use the number of necklaces smaller than $w$ ($RN(w)$), along with the other two sets ($RP(w)$ and $RE(w)$) to count the size of $RA(w)$.
- Note that the number of necklaces smaller than $w$ is equals the sum of:
    - 2 times the number of apalindromic bracelets not enclosing $w$.
    - The number of palindromic bracelets smaller than $w$.
    - The number of enclosing bracelets smaller than $w$.
- $RN(w) = 2RA(w) + RP(w) + RE(w)$.
- This summation can be rearranged to give: $RA(w) = \frac{1}{2}\left(RN(w) - (RP(w) + RE(w))\right)$.

# Counting $RP(w)$

- In order to count the number of palindromic bracelets smaller than $w$, it is useful to have a characterisation of palindromic bracelets.
- Recall that a word $w$ is palindromic if and only if $w = w^R$.
    - $aba = aba^R$.
- Any bracelet containing a palindromic word must be palindromic.
- However not all palindromic bracelets contain palindromic words
    - The bracelet $\{abab, baba\}$ is palindromic, however $abab \neq abab^R$ and $baba \neq baba^R$.

# Odd length palindromic bracelets

Lemma

*A bracelet b of odd length n is palindromic if and only if it contains* exactly *one unique word of the form $\phi x \phi^R$ for some word $\phi$ and symbol x.*

- Note that any word $w = \phi x \phi^r = w_1 w_2 \dots w_{(n-1)/2} x w_{(n-1)/2} \dots w_2 w_1$ is palindromic.

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- To show that $b$ must contain at least 1 palindromic word, let $|b|$ be the number of words in $b$.
    - Note that $|b|$ must be odd.

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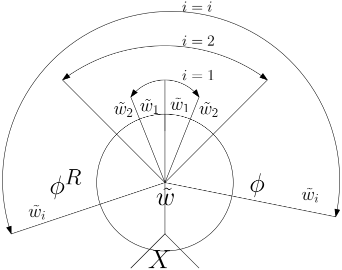
- To show that $b$ must contain at least 1 palindromic word, let $|b|$ be the number of words in $b$.
    - Note that $|b|$ must be odd.
- Let $a$ be the set of apalindromic words in $b$.
- Note that if $v \in a$ then $v^R \in a$ and $v \neq v^R$.
- As $|a|$ must be even, $|b| - |a| \geq 1$.

# Counting the number of odd length palindromic bracelets

- **Main Idea:** The main idea is to use the uniqueness of the palindromic representations to count the number of bracelets smaller than $w$.
- This is done by counting the number of unique prefixes of the form $\phi x \phi^R$ iteratively.



# Counting the number of odd length palindromic bracelets

- Explicitly generating the whole prefix tree is impractical.
- Instead, we look at sets $\mathrm{PO}(w, i, j, s)$ containing every word $u$ where:
    - $|u| = i$
    - $j$ is the longest suffix of $u$ that is a prefix of $w$.
    - $s$ is the lexicographically greatest subword of $w$ that is less than or equal to $u$.

| | |
|---|---|
| $\mathrm{PO}(abbcabbc, 3, 0, cab)$ | $ccc, ccb, cbc, cac$ |
| $\mathrm{PO}(abbcabbc, 3, 1, cab)$ | $cca, cba$ |
| $\mathrm{PO}(abbcabbc, 3, 2, cab)$ | $cab$ |

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| $\mathrm{PO}(abbcabbc, 3, 2, cab)$ | $c\mathbf{ab}$ |

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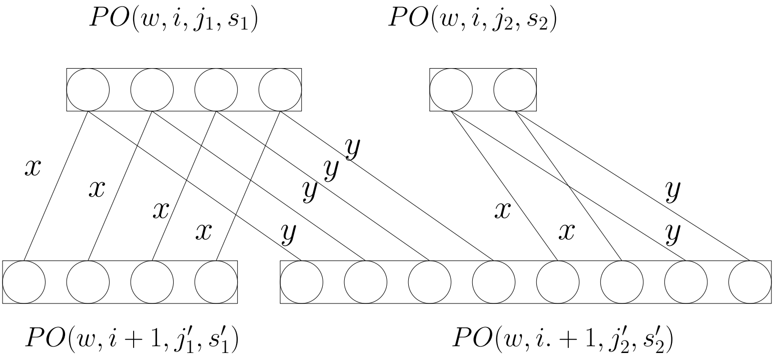
# Computing the size of PO($w, i, j, s$)



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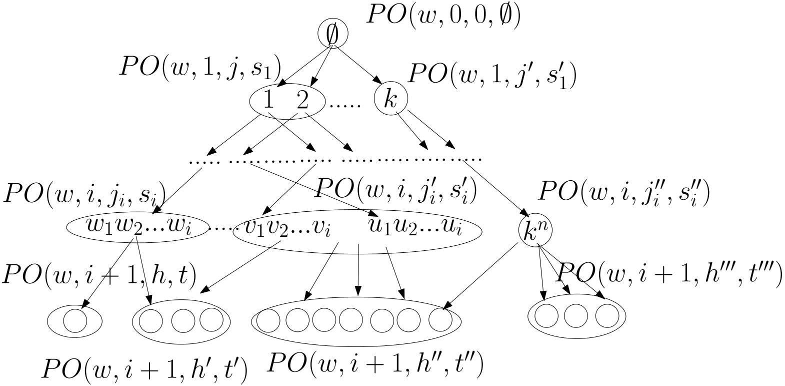

# Even Length palindromic bracelets

Lemma

*A bracelet $b$ of even length $n$ is palindromic if and only if it contains some word $w$ where either (1) $w = x\phi y\phi^R$ for symbols $x, y$ and word $\phi$ of length $\frac{n}{2} - 1$, or (2) $w = \phi\phi^R$ where $\phi$ is a word of length $\frac{n}{2}$.*

- The number of such words is counted in the same manner as in the odd case.
- The main difference is that these words are not unique, adding additional complexity to computing the number of palindromic bracelets of even length.

# Computing $RE(w)$

- Similarly to the Palindromic case, the first step in ranking the number of enclosing is defining a combinatorial structure to capture them.
- The canonical representation of every bracelet enclosing $w$ can be written in the form

$$w_1 w_2 \dots w_i x \phi$$

  where:
  - $w_1 w_2 \dots w_i$ is the prefix of $w$ of length $i$.
  - $x < w_{i+1}$
  - $\phi$ is a word such that every suffix of $x w_i w_{i-1} \dots w_1 \phi^R x w_i w_{i-1} \dots w_1$ starting at some symbol in $\phi$ is strictly greater than $w$.

# Counting Enclosing Bracelets

- Using the combinatorial characterisation, the number of enclosing words is counted by determining the number of possible values of $\phi$ for every pair $i \in 1 \dots n$ and $x \in \Sigma$.

$$\sum_{i=0}^{n-1} \sum_{x \in \Sigma} |\phi(w, i, x)|$$

- This is done in a recursive manner based on counting the number of words with no suffix greater than $w$ in increasing length.
- As each bracelet may only have a single such representation, by counting the number of representatives the exact number of enclosing bracelets can be counted.

# Unranking

- The inverse of the ranking process is unranking.
- This problem asks what the $i^{th}$ bracelet in the set $B(n, k)$ is.
- This problem was solved by Sawada and Williams [4] for Necklaces and Lyndon words in $O(n^3 \log k)$ time.
- We solve the unranking problem for bracelets in $O(n^5 \cdot k^2 \cdot \log(k))$ time.

# The Unranking Process

- To determine the $i^{th}$ bracelet, a binary search is performed using the ranking algorithm as a sub process.
- Note that the number of bracelets where the canonical representative has a prefix $\psi$ is given by $RB(\psi k^{n-|\psi|}) - RB(\psi^{n/|\psi|})$.
- Further, the rank of every bracelet with the prefix $\psi$ is between $RB(\psi^{n/|\psi|})$ and $RB(\psi k^{n-|\psi|})$.

# The Unranking Process

- The first symbol of $b$ must be the symbol $x$ for which $RB(x^n) \leq i \leq RB(xk^{n-1})$.
- This is found by performing a binary search over the alphabet.
- Similarly, the $2^{nd}$ symbol is the symbol $y$ where $RB((xy)^{n/2}) \leq i \leq RB((xy)k^{n-2})$.

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- Similarly, the $2^{nd}$ symbol is the symbol $y$ where $RB((xy)^{n/2}) \leq i \leq RB((xy)k^{n-2})$.
- Let $\psi$ be the prefix of $b$ of length $j - 1$.
- The $j^{th}$ symbol of $b$ is the symbol $x$ such that $RB((\psi x)^{n/j}) \leq i \leq RB(\psi xk^{n-j})$.
- This can also be found by a binary search.

# Conclusion

- We have presented a $O(n^4 \cdot k^2)$ time algorithm to rank bracelets.
- This is complimented by a $O(n^5 \cdot k^2 \cdot \log k)$ time unranking algorithm.
- In order to rank bracelets in polynomial time, we have also developed algorithms to rank both Palindromic and Enclosing bracelets in $O(n^3 \cdot k \cdot \log^2(n))$ and $O(n^4 \cdot k^2)$ time respectively.
- These algorithms may be used to rank the aperiodic counterparts to Bracelets, Palindromic Bracelets, and Enclosing bracelets at an additional factor of $O(n)$ operations.

# Open Problems

- Is there a $O(n^3)$ or faster algorithm for ranking Bracelets (either with a fixed alphabet or in general)?
- Can fixed density bracelets be ranked in polynomial time, and if so does there exist an $O(n^5)$ time algorithm to do so?

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