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COMPLEXITY OF MODAL LOGIC

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1   INTRODUCTION

This chapter is a basic introduction to the field of computational complexity in modal logic. We are mostly concerned with the following question: given a formula A and a set of formulas C, does there exists a model in which all of C is true at every world and A is true at some world? In other words, is $C \models \neg A$ or $C \not\models \neg A$ the case? This is the complement of the (global) consequence problem: $C \models A$ (is A true in every model in which all of C is true at every world). The special case of the consequence problem in which C is the empty set is called the validity problem, and its complement is the satisfiability problem. For finite C, the local consequence problem reduces to the validity problem, because of the deduction theorem.

For many modal logics, these problems are decidable. Here we look at the *difficulty* of deciding them. This is the topic of the theory of computational complexity. As Wikipedia has it:

> Computational complexity theory is part of the theory of computation dealing with the resources required during computation to solve a given problem. The most common resources are time (how many steps it takes to solve a problem) and space (how much memory it takes). Complexity theory differs from computability theory, which deals with whether a problem can be solved at all, regardless of the resources required.

Standard references to this field are [27] and [32].

Organization. The current section introduces common decision problems in modal logic and derives three useful properties of modal logics. In Section 2 we discuss the basic methods of establishing decidability and complexity results for the satisfiability problem in modal logic. In Section 3 we review the basic notions of computational complexity theory and after that we reduce several tiling problems to modal satisfiability problems in order to obtain lower complexity bounds. These say roughly that —up to a polynomial— one cannot give a better algorithm for the problem at hand. Throughout the text, we hardly give references. We end with some historical notes.

Links to Wikipedia. This chapter contains a lot of terminology with which the average logician might not be familiar. We have used links to the relevant Wikipedia entries to facilitate the reader. When viewing this document in a PDF reader, clicking on the highlighted terms should open the relevant Wikipedia page in a browser.

1.1 Examples of decision problems in modal logic

This chapter is about solving problems in modal logic. What is a problem? A problem for us is a yes/no question. These problems are typically formalized as set-membership problems. Here are some examples.

Suppose first that a logic L is presented as a set of wffs (well formed formulas), as in Chapter 2 of this handbook. Then membership in L is the same as being valid. Thus the validity problem equals the L membership problem.

Alternatively, we can define a modal logic as a triple (Wffs, Struc, $\models$), a set of wffs, a class of models, and a relation between the two. (To be precise, $\models$ is a relation between a model $\mathcal{M}$, a world w and a wff A.) This is essentially the way logics are defined in abstract model theory; see Chapter 1 of this handbook for further discussion.

In this richer setting, more natural decision problems show up:

Model checking

1. Given a finite model $\mathcal{M}$, is $\mathcal{M}$ a member of Struc?
2. Given a finite model $\mathcal{M}$ in Struc, a world w in $\mathcal{M}$ and a formula $A \in$ Wffs, does $\mathcal{M}, w \models A$ hold?

Satisfiability Given a formula $A \in$ Wffs, does there exists a model $\mathcal{M}$ in Struc and a world w in $\mathcal{M}$ such that $\mathcal{M}, w \models A$ hold?