An Automata-Theoretic Approach to Uniform Interpolation and Approximation in the Description Logic $\mathcal{EL}$

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Abstract

We study (i) uniform interpolation for TBoxes that are formulated in the lightweight description logic $\mathcal{EL}$ and (ii) $\mathcal{EL}$-approximations of TBoxes formulated in more expressive languages. In both cases, we give model-theoretic characterizations based on simulations and cartesian products, and we develop algorithms that decide whether interpolants and approximants exist. We present a uniform approach to both problems, based on a novel amorphous automaton model called $\mathcal{EL}$ automata (EA). Using EAs, we also establish a simpler proof of the known result that conservative extensions of $\mathcal{EL}$-TBoxes can be decided in EXPTIME.

1 Introduction

Formal ontologies provide a conceptual model of a domain of interest by describing the vocabulary of that domain in terms of a logical language, such as a description logic (DL). To cater for different applications and uses of ontologies, DLs and other ontology languages vary significantly regarding expressive power and computational complexity (Baader et al. 2003). For example, lightweight DLs such as the OWL2 profile OWL2EL and its underlying core logic $\mathcal{EL}$ are relatively inexpressive, focussing mainly on conjunction and existential quantification, but admit PTIME reasoning. In contrast, expressive DLs such as OWL2DL, $\mathcal{ALC}$, and $\mathcal{SHIQ}$ are equipped with all Boolean operators and existential as well as universal quantification, and may additionally include a variety of other features; consequently, the complexity of reasoning is between EXPTIME and 2NEXPTIME.

When an ontology is used for a new purpose, it is often desirable or even unavoidable to *customize* the ontology in a suitable way. In this paper, we look at two rather common such customizations: (i) restricting the signature (set of vocabulary items) that is covered by the ontology and (ii) restricting the logical language that the ontology is formulated in.

The basic operation for restricting the signature of an ontology is uniform interpolation, also called forgetting and variable elimination (Reiter and Lin 1994; Lang, Liberatore, and Marquis 2003; Eiter et al. 2006; Konev, Walther, and Wolter 2009; Kontchakov, Wolter, and Zakharyaschev 2010; Wang et al. 2010a). Specifically, uniform interpolation is the problem of constructing, given a TBox $\mathcal{T}$ and a signature Σ, a new ontology $\mathcal{T}'$ that uses only symbols from Σ and has the same logical consequences as $\mathcal{T}$ as far as the signature Σ is concerned. In other words, $\mathcal{T}'$ is obtained from $\mathcal{T}$ by forgetting all non-Σ-symbols. We then call $\mathcal{T}'$ a *uniform Σ-interpolant* of $\mathcal{T}$. There are various applications of uniform interpolation, of which we mention three. *Ontology reuse*. When reusing an existing ontology in a new application, then typically only a small number of the symbols is relevant. Instead of reusing the whole ontology, one can thus use the potentially much smaller ontology that results from forgetting the extraneous symbols. *Predicate hiding*. When an ontology is to be published, but some part of it has to be concealed from the public, then this part can be removed by forgetting the symbols that belong to it (Grau and Motik 2010). *Ontology summary.* The result of forgetting often provides a smaller and more focussed ontology that summarizes what the original ontology says about the retained symbols, potentially facilitating ontology comprehension.

The basic operation for restricting the logical language of an ontology is approximation: given an ontology $\mathcal{T}$ formulated in some expressive DL $\mathcal{L}$, construct a new ontology $\mathcal{T}'$ in a less expressive DL $\mathcal{L}'$ such that $\mathcal{T}'$ is logically entailed by $\mathcal{T}$ and is most specific with this property, i.e., any $\mathcal{L}'$-ontology $\mathcal{T}''$ entailed by $\mathcal{T}$ is also entailed by $\mathcal{T}'$. We then call $\mathcal{T}'$ an *$\mathcal{L}'$-approximant of* $\mathcal{T}$. An introduction to this type of *semantic* approximation is given in (Selman and Kautz 1996). We mention two relevant applications of approximation. *Ontology reuse*. Approximation is required when an ontology that is formulated in a DL $\mathcal{L}$ is reused in an application which requires reasoning, but where reasoners for $\mathcal{L}$ are not available or not sufficiently efficient. For example, if ontologies are used to access instance data, then scalable query answering is currently only available for lightweight DLs such as $\mathcal{EL}$, but not for more expressive ones such as $\mathcal{ALC}$. Indeed, we show that the uniform interpolants and approximants studied in this paper can both be exploited for answering queries over instance data. *Ontology summary*. The comprehension of an ontology may be hindered not only by a too large vocabulary, but also by a too complex and detailed modeling. To get to grips with understanding an ontology, it can thus be useful to approximate it in a less expressive DL, in this way concealing all modeling details that can only be expressed in more powerful languages.

A general problem with both uniform interpolation and approximation is that the result of these operations need not be expressible in the desired language. For example, the uniform Σ-interpolant of the $\mathcal{EL}$-TBox

$$\{A \sqsubseteq \exists r.B, B \sqsubseteq \exists r.B\}$$

with $\Sigma = \{A, r\}$ is not expressible as a (finite) $\mathcal{EL}$-TBox, and the $\mathcal{EL}$-approximant of the $\mathcal{ELU}$-TBox

$$\{A \sqsubseteq \exists r.(B_1 \sqcup B_2), B_1 \sqcup B_2 \sqsubseteq \exists r.(B_1 \sqcup B_2)\}$$

is not expressible as a (finite) $\mathcal{EL}$-TBox either. Thus, when working with uniform interpolation and approximation, a fundamental task is to determine whether the desired result exists in the first place.

In this paper, we concentrate on the description logic $\mathcal{EL}$ and consider (i) uniform interpolants of $\mathcal{EL}$-TBoxes and (ii) $\mathcal{EL}$-approximants of TBoxes formulated in more expressive languages, with an emphasis on the extension $\mathcal{ELU}$ of $\mathcal{EL}$ with disjunction. Our main aims are, on the one hand, to provide semantic characterizations of uniform interpolants and approximants, based on model-theoretic notions such as (equi)simulations and products. On the other hand, we develop algorithms for deciding the existence of uniform interpolants and approximants and analyze the computational complexity of these problems. While we do not directly study the actual computation of uniform interpolants and approximants, we believe that the machinery developed in this paper provides an important technical foundation also for this task.

We use a uniform approach that allows us to decide the existence of both uniform interpolants and $\mathcal{EL}$-approximants and highlights the commonalities and differences between uniform interpolation and approximation. Our main technical tool is a novel type of automaton, called $\mathcal{EL}$ automaton (EA), which bears some similarity to the 'amorphous' automata models introduced in (Janin and Walukiewicz 1995; Wilke 2001). In particular, EAs are tree automata in spirit, but run directly on (not necessarily tree-shaped) DL interpretations. In contrast to existing automata models, they are tailored towards Horn-like logics and, in particular, $\mathcal{EL}$-TBoxes. Consequently, the languages that they accept are closed under intersection and projection, but not under union and complementation. While the expressive power of EAs is only moderately larger than that of $\mathcal{EL}$-TBoxes, we show that it is *always* possible to express uniform interpolants of $\mathcal{EL}$-TBoxes and $\mathcal{EL}$-approximants of classical $\mathcal{ELU}$-TBoxes as EAs (a classical TBox is a set of statements $A \equiv C$ and $A \sqsubseteq C$; definitorial cycles are allowed). This enables us to decompose the aforementioned existence problems into two separate steps: first compute the desired uniform interpolant or approximant, represented as an EA, and then decide whether the EA accepts a language that can be defined using an $\mathcal{EL}$-TBox. Note that the latter step is a pure EA problem and does not involve reference to uniform interpolation or approximation. Using this machinery, we show that deciding the existence of uniform interpolants of $\mathcal{EL}$-TBoxes is EXPTIME-complete and deciding the existence of $\mathcal{EL}$-approximants of classical $\mathcal{ELU}$-TBoxes is in 2EXPTIME (and EXPTIME-hard). The precise complexity of the latter problem remains open. We also use EAs to give a new and arguably simpler proof of the known result that deciding conservative extensions of $\mathcal{EL}$-TBoxes is in EXPTIME (Lutz and Wolter 2010).

The existence problems considered in this paper are known to be challenging. For the expressive DL $\mathcal{ALC}$, a number of results for uniform interpolation were obtained in (Wang et al. 2009; 2010b), but the first correct algorithm for deciding the existence of uniform interpolants for TBoxes was only recently given in (Lutz and Wolter 2011). The first paper concerned with uniform interpolation in $\mathcal{EL}$ is (Konev, Walther, and Wolter 2009), but unlike the current paper it considers only *classical* $\mathcal{EL}$-TBoxes instead of general ones, and it does not prove any decidability results for the existence of uniform interpolants. A method for computing interpolants of general $\mathcal{EL}$-TBoxes and deciding their existence in EXPTIME was recently claimed in (Nikitina 2011), but while the approach is promising we found it difficult to fully verify the claimed results based on the currently available material. Approximation has been considered on the level of concepts in (Brandt, Küsters, and Turhan 2002). In contrast, no results appear to be known for deciding the existence of approximants on the TBox level, with the notable exception of DL-Lite (Botoeva, Calvanese, and Rodriguez-Muro 2010). Instead, research has focussed on heuristic (and incomplete) approaches to computing TBox approximants, see for example (Pan and Thomas 2007; Ren, Pan, and Zhao 2010). While the results for $\mathcal{EL}$-approximation of classical $\mathcal{ELU}$-TBoxes obtained in this paper may not seem too impressive on first sight, already this restricted case requires intricate technical machinery and we view it as an important first step towards sound and complete TBox approximation beyond DL-Lite.

Most proofs are deferred to the long version of this paper available at http://www.csc.liv.ac.uk/~frank/publ/publ.html.

2 Preliminaries

Let $\mathsf{N_C}$ and $\mathsf{N_R}$ be countably infinite and mutually disjoint sets of *concept* and *role names*. *$\mathcal{EL}$-concepts* C are built according to the rule

$$C \quad := \quad A \quad | \quad \top \quad | \quad \bot \quad | \quad C \sqcap D \quad | \quad \exists r.C$$

where A ranges over $\mathsf{N_C}$, r over $\mathsf{N_R}$, and C, D over $\mathcal{EL}$-concepts.[1] An *$\mathcal{EL}$-concept inclusion (CI)* is an expression $C \sqsubseteq D$ with C, D $\mathcal{EL}$-concepts, and a *(general) $\mathcal{EL}$-TBox* is a finite set of $\mathcal{EL}$-CIs. In some cases, we drop the finiteness condition on TBoxes and then explicitly speak of *infinite TBoxes*.

The semantics of $\mathcal{EL}$ is given by *interpretations* $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$, where the *domain* $\Delta^\mathcal{I}$ is a non-empty set and $\cdot^\mathcal{I}$ is an *interpretation function* that maps each concept name

[1]Note that $\mathcal{EL}$ is sometimes defined without "$\bot$". The results in this paper are independent of whether or not $\bot$ is present.

A to a subset $A^{\mathcal{I}}$ of $\Delta^{\mathcal{I}}$ and each role name r to a binary relation $r^{\mathcal{I}}$ over $\Delta^{\mathcal{I}}$. We extend $\cdot^{\mathcal{I}}$ as follows:

$$\begin{array}{rclrcl} \top^{\mathcal{I}} & := & \Delta^{\mathcal{I}} & \bot^{\mathcal{I}} & := & \emptyset \\ (C \sqcap D)^{\mathcal{I}} & := & C^{\mathcal{I}} \cap D^{\mathcal{I}} & & & \\ (\exists r.C)^{\mathcal{I}} & := & \{d \in \Delta^{\mathcal{I}} \mid \exists e \in C^{\mathcal{I}} : (d,e) \in r^{\mathcal{I}}\} & & & \end{array}$$

An interpretation $\mathcal{I}$ *satisfies* a CI $C \sqsubseteq D$, written $\mathcal{I} \models C \sqsubseteq D$, if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$; $\mathcal{I}$ is a *model* of a TBox $\mathcal{T}$ if $\mathcal{I}$ satisfies all CIs in $\mathcal{T}$. We use $\mathsf{mod}(\mathcal{T})$ to denote the class of all models of $\mathcal{T}$ and write $\mathcal{T} \models C \sqsubseteq D$ if every model of $\mathcal{T}$ satisfies $C \sqsubseteq D$ and $\mathcal{T} \models \mathcal{T}'$ if $\mathcal{T} \models C \sqsubseteq D$ for all $C \sqsubseteq D \in \mathcal{T}'$.

A *signature* is a set $\Sigma \subseteq \mathsf{N_C} \cup \mathsf{N_R}$ of concept and role names, which we uniformly call *symbols* in this context. The *signature* $\mathsf{sig}(C)$ *of a concept* C is the set of symbols that occur in C, and likewise for $\mathsf{sig}(C \sqsubseteq D)$ and $\mathsf{sig}(\mathcal{T})$ for CIs $C \sqsubseteq D$ and TBoxes $\mathcal{T}$. An $\mathcal{EL}_\Sigma$*-concept* is an $\mathcal{EL}$-concept C with $\mathsf{sig}(C) \subseteq \Sigma$ and likewise for $\mathcal{EL}_\Sigma$*-CIs* and $\mathcal{EL}_\Sigma$*-TBoxes*.

We sometimes use TBoxes formulated in more expressive logics than $\mathcal{EL}$, including full first-order logic (FO). More specifically, an *FO-TBox* is a finite set of FO sentences using unary predicates from $\mathsf{N_C}$ and binary predicates from $\mathsf{N_R}$ (plus equality), and without any function symbols or constants. It is well-known that any $\mathcal{EL}$-TBox can be viewed as an FO-TBox, and the same is true for many other DLs such as $\mathcal{ELU}$ and $\mathcal{ALC}$ (see (Baader et al. 2003) for more details).

The definitions of our main notions, uniform interpolation and approximation, which are given in the two subsequent sections, both rely on the logical consequences of a TBox. In the case of uniform interpolation, there is an emphasis on the signature in which such consequences are formulated while approximation emphasises the logic in which they are formulated. We treat these different aspects in a uniform way based on the notions of $\mathcal{EL}_\Sigma$-entailment and $\mathcal{EL}_\Sigma$-inseparability. Let Σ be a signature. An FO-TBox $\mathcal{T}$ $\mathcal{EL}_\Sigma$*-entails* an FO-TBox $\mathcal{T}'$ if for all $\mathcal{EL}_\Sigma$-CIs $C \sqsubseteq D$, $\mathcal{T}' \models C \sqsubseteq D$ implies $\mathcal{T} \models C \sqsubseteq D$. Note that when $\Sigma = \mathsf{N_C} \cup \mathsf{N_R}$ and $\mathcal{T}'$ is an $\mathcal{EL}$-TBox, then $\mathcal{T}$ $\mathcal{EL}_\Sigma$-entails $\mathcal{T}'$ if and only if $\mathcal{T} \models \mathcal{T}'$. Two FO-TBoxes $\mathcal{T}$ and $\mathcal{T}'$ are $\mathcal{EL}_\Sigma$*-inseparable*, written $\mathcal{T}_1 \equiv_\Sigma^{\mathcal{EL}} \mathcal{T}_2$, if $\mathcal{T}$ $\mathcal{EL}_\Sigma$-entails $\mathcal{T}'$ and vice versa. If $\mathcal{T}$ and $\mathcal{T}'$ are $\mathcal{EL}$-TBoxes and $\Sigma = \mathsf{N_C} \cup \mathsf{N_R}$, then $\mathcal{T}_1$ and $\mathcal{T}_2$ are logically equivalent if they are $\mathcal{EL}_\Sigma$-inseparable. In all these notations, we drop Σ when $\Sigma = \mathsf{N_C} \cup \mathsf{N_R}$.

3 Uniform Interpolation and Conservative Extensions

We introduce uniform interpolation and conservative extensions and establish semantic characterisations based on a certain kind of simulation between interpretations.

Definition 1. For $\mathcal{EL}$-TBoxes $\mathcal{T}$ and $\mathcal{T}'$, we say that

- $\mathcal{T}$ is a *uniform* $\mathcal{EL}_\Sigma$*-interpolant* of $\mathcal{T}'$ if $\mathsf{sig}(\mathcal{T}) \subseteq \Sigma \subseteq \mathsf{sig}(\mathcal{T}')$, $\mathcal{T}' \models \mathcal{T}$, and $\mathcal{T}$ $\mathcal{EL}_\Sigma$-entails $\mathcal{T}'$;
- $\mathcal{T}'$ is an $\mathcal{EL}$-conservative extension of $\mathcal{T}$ if $\mathcal{T} \subseteq \mathcal{T}'$ and $\mathcal{T}$ $\mathcal{EL}_\Sigma$-entails $\mathcal{T}'$ for $\Sigma = \mathsf{sig}(\mathcal{T})$.

Both notions can be characterized using $\mathcal{EL}_\Sigma$-inseparability: it is not hard to verify that an $\mathcal{EL}_\Sigma$-TBox $\mathcal{T}$ is a uniform $\mathcal{EL}_\Sigma$-interpolant of an $\mathcal{EL}$-TBox $\mathcal{T}'$ if and only if $\mathcal{T} \equiv_\Sigma^{\mathcal{EL}} \mathcal{T}'$, and $\mathcal{T}'$ is an $\mathcal{EL}$-conservative extension of $\mathcal{T}$ if and only if $\mathcal{T} \subseteq \mathcal{T}'$ and $\mathcal{T} \equiv_\Sigma^{\mathcal{EL}} \mathcal{T}'$. Also note that uniform $\mathcal{EL}_\Sigma$-interpolants are unique up to logical equivalence, when they exist.

Forgetting is dual to uniform interpolation in the following sense: a TBox $\mathcal{T}'$ is *the result of forgetting about a signature* Σ *in a TBox* $\mathcal{T}$ if $\mathcal{T}'$ is a uniform $\mathsf{sig}(\mathcal{T}) \setminus \Sigma$-interpolant of $\mathcal{T}$. Therefore, the results presented in this paper for uniform interpolation also apply to forgetting. For more information, see e.g. (Wang et al. 2010b; Lutz and Wolter 2011).

As an example, consider the following TBox $\mathcal{T}_1$:

$$\begin{array}{rcl} \mathsf{Patient} \sqcap \exists \mathsf{finding}.\mathsf{MajorFinding} & \sqsubseteq & \mathsf{InPatient} \\ \mathsf{Finding} \sqcap \exists \mathsf{status}.\mathsf{PotentiallyLethal} & \sqsubseteq & \mathsf{MajorFinding} \\ \mathsf{MajorFinding} & \sqsubseteq & \mathsf{Finding} \end{array}$$

For $\Sigma = \mathsf{sig}(\mathcal{T}_1) \setminus \{\mathsf{MajorFinding}\}$, a uniform $\mathcal{EL}_\Sigma$-interpolant of $\mathcal{T}_1$ (equivalently: the result of forgetting $\mathsf{MajorFinding}$) consists of the single CI

$$\mathsf{Patient} \sqcap \exists \mathsf{finding}.(\mathsf{Finding} \sqcap \exists \mathsf{status}.\mathsf{PotentiallyLethal}) \sqsubseteq \mathsf{InPatient}.$$

As another example, consider the TBox

$$\mathcal{T}_2 = \{A \sqsubseteq \exists r.B, B \sqsubseteq \exists r.B\}$$

mentioned in the introduction and let $\Sigma = \{A, r\}$. It can be shown that the infinite TBox $\mathcal{T}' = \{A \sqsubseteq \exists r^n.\top \mid n \geq 1\}$ satisfies $\mathcal{T}_2 \equiv_\Sigma^{\mathcal{EL}} \mathcal{T}'$ and thus, except for its infinity, qualifies as a uniform $\mathcal{EL}_\Sigma$-interpolant of $\mathcal{T}_2$. However, uniform interpolants are required to be *finite* TBoxes and, in this case, it can be proved that there is no (finite) uniform $\mathcal{EL}_\Sigma$-interpolant of $\mathcal{T}_2$ (see appendix of the long version of this paper).

Note that uniform $\mathcal{EL}_\Sigma$-interpolants are weaker than uniform $\mathcal{ALC}_\Sigma$-interpolants as studied in (Lutz and Wolter 2011), even when the original TBox $\mathcal{T}$ is formulated in $\mathcal{EL}$. The reason is that $\mathcal{ALC}_\Sigma$-interpolants are required to preserve all $\mathcal{ALC}$-consequences of $\mathcal{T}$ formulated in Σ, instead of all $\mathcal{EL}$-consequences. In fact, we show in the appendix of the long version that there is an $\mathcal{EL}$-TBox that has a uniform $\mathcal{EL}_\Sigma$-interpolant, but no uniform $\mathcal{ALC}_\Sigma$-interpolant. It follows that the results and algorithms in (Lutz and Wolter 2011) do not apply to the framework studied in this paper.

We have defined uniform interpolants based on subsumption. To illustrate their utility beyond subsumption, we show that they can be used for instance query answering. DL instance data is represented by an *ABox*, which is a finite set of assertions of the form $A(a)$ and $r(a,b)$ where $A \in \mathsf{N_C}$, $r \in \mathsf{N_R}$, and a, b are *individual names*. An *instance query* takes the form $C(a)$, where C is an $\mathcal{EL}$-concept and a an individual name; we speak of a Σ*-instance query* when C is an $\mathcal{EL}_\Sigma$-concept and write $\mathcal{A}, \mathcal{T} \models C(a)$ if every model of $\mathcal{A}$ and $\mathcal{T}$ satisfies $C(a)$, see e.g. (Baader et al. 2003) for more details. *Instance query answering* is the problem to decide, given an ABox $\mathcal{A}$, TBox $\mathcal{T}$, and instance query $C(a)$, whether $\mathcal{A}, \mathcal{T} \models C(a)$. The following result, proved in (Lutz and Wolter 2010) in terms of $\mathcal{EL}_\Sigma$-inseparability, demonstrates the utility of uniform interpolants for instance query answering: when only a few of the symbols in a TBox $\mathcal{T}$ are relevant for instance query answering, then $\mathcal{T}$ can be replaced with a potentially much smaller uniform interpolant.

Theorem 2. *For any $\mathcal{EL}$-TBox $\mathcal{T}$, signature Σ, and $\mathcal{EL}_\Sigma$-TBox $\mathcal{T}'$, $\mathcal{T}'$ is a uniform $\mathcal{EL}_\Sigma$-interpolant of $\mathcal{T}$ iff for all Σ-ABoxes $\mathcal{A}$ and Σ-instance queries $C(a)$, $\mathcal{A}, \mathcal{T} \models C(a)$ iff $\mathcal{A}, \mathcal{T}' \models C(a)$.*

For the announced semantic characterization, we introduce simulations. A *pointed interpretation* is a pair $(\mathcal{I}, d)$ with $\mathcal{I}$ an interpretation and $d \in \Delta^\mathcal{I}$.

Definition 3. Let Σ be a signature and $(\mathcal{I}_1, d_1)$, $(\mathcal{I}_2, d_2)$ be pointed interpretations. A relation $S \subseteq \Delta^{\mathcal{I}_1} \times \Delta^{\mathcal{I}_2}$ is a Σ-*simulation between* $(\mathcal{I}_1, d_1)$ *and* $(\mathcal{I}_2, d_2)$, written $S : (\mathcal{I}_1, d_1) \leq_\Sigma (\mathcal{I}_2, d_2)$, if $(d_1, d_2) \in S$ and the following conditions hold:

(base) for all $A \in \Sigma \cap \mathsf{N_C}$ and $(e_1, e_2) \in S$, if $e_1 \in A^{\mathcal{I}_1}$ then $e_2 \in A^{\mathcal{I}_2}$;

(forth) for all $r \in \Sigma \cap \mathsf{N_R}$, $(e_1, e_2) \in S$, and $(e_1, e_1') \in r^{\mathcal{I}_1}$, there is an $(e_2, e_2') \in r^{\mathcal{I}_2}$ with $(e_1', e_2') \in S$.

If such an S exists, then we write $(\mathcal{I}_1, d_1) \leq_\Sigma (\mathcal{I}_2, d_2)$. We say that $(\mathcal{I}_1, d_1)$ *and* $(\mathcal{I}_2, d_2)$ *are* Σ-*equisimilar* if $(\mathcal{I}_1, d_1) \leq_\Sigma (\mathcal{I}_2, d_2)$ and $(\mathcal{I}_2, d_2) \leq_\Sigma (\mathcal{I}_1, d_1)$, written $(\mathcal{I}_1, d_1) \approx_\Sigma (\mathcal{I}_2, d_2)$.

There is a close relationship between (equi)simulations and the truth of $\mathcal{EL}$-concepts in pointed interpretations, see (Lutz and Wolter 2010). To make this precise, we say that $(\mathcal{I}_1, d_1)$ and $(\mathcal{I}_2, d_2)$ are $\mathcal{EL}_\Sigma$-*equivalent*, written $(\mathcal{I}_1, d_1) \equiv_\Sigma^{\mathcal{EL}} (\mathcal{I}_2, d_2)$, if for all $\mathcal{EL}_\Sigma$-concepts C, we have $d_1 \in C^{\mathcal{I}_1}$ iff $d_2 \in C^{\mathcal{I}_2}$. An interpretation $\mathcal{I}$ has *finite outdegree* if $\{d' \mid (d, d') \in \bigcup_{r \in \mathsf{N_R}} r^\mathcal{I}\}$ is finite, for all $d \in \Delta^\mathcal{I}$.

Lemma 4. *For all pointed interpretations $(\mathcal{I}_1, d_1)$, $(\mathcal{I}_2, d_2)$ and signatures Σ, $(\mathcal{I}_1, d_1) \approx_\Sigma (\mathcal{I}_2, d_2)$ implies $(\mathcal{I}_1, d_1) \equiv_\Sigma^{\mathcal{EL}} (\mathcal{I}_2, d_2)$. The converse holds if $\mathcal{I}_1, \mathcal{I}_2$ are of finite outdegree.*

Now for the semantic characterizations. For a class $\mathfrak{C}$ of interpretations and a signature Σ, we use $\mathsf{cl}_\approx^\Sigma(\mathfrak{C})$ to denote the *closure under global Σ-equisimulations* of $\mathfrak{C}$, i.e., the class of all interpretations $\mathcal{I}$ such that for all $d \in \Delta^\mathcal{I}$, there is a $\mathcal{J} \in \mathfrak{C}$ and a $d' \in \Delta^\mathcal{J}$ such that $(\mathcal{I}, d) \approx_\Sigma (\mathcal{J}, d')$. When $\Sigma = \mathsf{N_C} \cup \mathsf{N_R}$, we simply write $\mathsf{cl}_\approx(\mathfrak{C})$ instead of $\mathsf{cl}_\approx^\Sigma(\mathfrak{C})$. It was shown in (Lutz, Piro, and Wolter 2011) that for each $\mathcal{EL}$-TBox $\mathcal{T}$, $\mathsf{mod}(\mathcal{T})$ is closed under global equisimulations.

Theorem 5. *Let $\mathcal{T}$, $\mathcal{T}'$ be $\mathcal{EL}$-TBoxes and Σ a signature.*

1. *$\mathcal{T}$ $\mathcal{EL}_\Sigma$-entails $\mathcal{T}'$ iff $\mathsf{mod}(\mathcal{T}) \subseteq \mathsf{cl}_\approx^\Sigma(\mathsf{mod}(\mathcal{T}'))$;*
2. *Let $\mathcal{T} \subseteq \mathcal{T}'$. Then $\mathcal{T}'$ is an $\mathcal{EL}$-conservative extension of $\mathcal{T}$ iff $\mathsf{mod}(\mathcal{T}) \subseteq \mathsf{cl}_\approx^\Sigma(\mathsf{mod}(\mathcal{T}'))$ for $\Sigma = \mathsf{sig}(\mathcal{T})$;*
3. *Let $\mathsf{sig}(\mathcal{T}) \subseteq \Sigma \subseteq \mathsf{sig}(\mathcal{T}')$. Then $\mathcal{T}$ is a uniform $\mathcal{EL}_\Sigma$-interpolant of $\mathcal{T}'$ iff $\mathsf{mod}(\mathcal{T}) = \mathsf{cl}_\approx^\Sigma(\mathsf{mod}(\mathcal{T}'))$.*

Proof. (Sketch) The challenging part is to establish Point 1. Once this is done, Points 2 and 3 follow by definition of conservative extensions/uniform interpolants and by Lemma 4. In the proof of Point 1, the "$\Leftarrow$" direction essentially consists of an application of Lemma 4: assume $\mathsf{mod}(\mathcal{T}) \subseteq \mathsf{cl}_\approx^\Sigma(\mathsf{mod}(\mathcal{T}'))$ and $\mathcal{T}' \models C \sqsubseteq D$ with $\mathsf{sig}(C), \mathsf{sig}(D) \subseteq \Sigma$, and to the contrary of what is to be shown, $\mathcal{T} \not\models C \sqsubseteq D$; then there is an $\mathcal{I} \in \mathsf{mod}(\mathcal{T})$ with $d \in C^\mathcal{I} \setminus D^\mathcal{I}$ and we have $\mathcal{I} \in \mathsf{cl}_\approx^\Sigma(\mathsf{mod}(\mathcal{T}'))$, which yields a model $\mathcal{J}$ of $\mathcal{T}'$ and a $d' \in \Delta^\mathcal{J}$ with $(\mathcal{I}, d) \approx_\Sigma (\mathcal{J}, d')$; by (the first part of) Lemma 4, $d' \in C^\mathcal{J} \setminus D^\mathcal{J}$ in contradiction to $\mathcal{J}$ being a model of $\mathcal{T}'$. The "$\Rightarrow$" direction can be proved applying (the second part of) Lemma 4, but only for interpretations $\mathcal{I}$ of finite outdegree. To obtain"$\Rightarrow$" in its full generality, we rely on automata-theoretic result obtained later on. The proof is given in Section 7. ❏

4 TBox Approximation

We introduce TBox approximation and establish a semantic characterization based on global equisimulations and products of interpretations. The characterization is rather general and applies to the $\mathcal{EL}$-approximation of FO-TBoxes.

Definition 6. Let $\mathcal{T}$ be an $\mathcal{EL}$-TBox and $\mathcal{T}'$ an FO-TBox. Then $\mathcal{T}$ is an $\mathcal{EL}$-*approximant of* $\mathcal{T}'$ if $\mathcal{T}' \models \mathcal{T}$ and for any $\mathcal{EL}$-TBox $\mathcal{T}''$ with $\mathcal{T}' \models \mathcal{T}''$, we have $\mathcal{T} \models \mathcal{T}''$.

It is easy to verify that an $\mathcal{EL}$-TBox $\mathcal{T}$ is an $\mathcal{EL}$-*approximant of* an FO-TBox $\mathcal{T}'$ if and only if $\mathcal{T} \equiv^{\mathcal{EL}} \mathcal{T}'$. Also, $\mathcal{EL}$-approximants of an FO-TBox are unique up to logical equivalence, if they exist. Note the similarity to the remark after Definition 1.

When establishing algorithms and complexity results for approximation, we will have a much more modest aim than the approximation of full FO-TBoxes. Specifically, we will consider the approximation of classical $\mathcal{ELU}$-TBoxes by general $\mathcal{EL}$-TBoxes. Here, $\mathcal{ELU}$ is the extension of $\mathcal{EL}$ with the union constructor $\sqcup$ whose semantics is $(C \sqcup D)^\mathcal{I} = C^\mathcal{I} \cup D^\mathcal{I}$, and a *classical TBox* is a finite set of $\mathcal{ELU}$-CIs of the form $A \sqsubseteq C$ or $A \equiv C$ where A is a concept name and left-hand sides of CIs are unique within the TBox. Subsumption is ExpTime-complete in the presence of classical $\mathcal{ELU}$-TBoxes while it is PTime-complete for general $\mathcal{EL}$-TBoxes, thus approximation pays off in terms of computational complexity. As an example, consider the $\mathcal{ELU}$-TBox $\mathcal{T}$ that consists of

$$\begin{aligned} \mathsf{Node} &\sqsubseteq \mathsf{Left} \sqcup \mathsf{Right} \\ \mathsf{Left} &\sqsubseteq \exists \mathsf{succ}.(\mathsf{Left} \sqcup \mathsf{Right}) \\ \mathsf{Right} &\sqsubseteq \exists \mathsf{succ}.(\mathsf{Left} \sqcup \mathsf{Right}) \end{aligned}$$

The infinite TBox

$$\mathcal{T}' = \{X \sqsubseteq \exists \mathsf{succ}^n.\top \mid X \in \{\mathsf{Node}, \mathsf{Left}, \mathsf{Right}\}, n \geq 0\}$$

satisfies $\mathcal{T} \equiv^{\mathcal{EL}} \mathcal{T}'$ and thus qualifies as an $\mathcal{EL}$-approximant except for its infinity. However, $\mathcal{EL}$-approximants need to be *finite* and it can be proved that there is no finite $\mathcal{EL}$-TBox with $\mathcal{T} \equiv^{\mathcal{EL}} \mathcal{T}'$, thus no $\mathcal{EL}$-approximant. If we add $\mathsf{Left} \sqsubseteq \mathsf{Node}$ and $\mathsf{Right} \sqsubseteq \mathsf{Node}$ to $\mathcal{T}$, then an $\mathcal{EL}$-approximant consists of these two CIs plus $\mathsf{Node} \sqsubseteq \exists \mathsf{succ}.\mathsf{Node}$.

Note that it makes sense to approximate classical $\mathcal{ELU}$-TBoxes by *general* $\mathcal{EL}$-TBoxes rather than classical ones. For example, the $\mathcal{EL}$-approximant of

$$\{A \equiv (B_1 \sqcup B_2) \sqcap (B'_1 \sqcup B'_2)\}$$

consists of the CIs $B_i \sqcap B'_j \sqsubseteq A$, $i, j \in \{1, 2\}$, whereas its approximation as a classical $\mathcal{EL}$-TBox is empty.

Just like uniform $\mathcal{EL}_\Sigma$-interpolants, $\mathcal{EL}$-approximations of classical $\mathcal{ELU}$-TBoxes are useful for answering queries over instance data. In particular, answering instance queries in the presence of classical $\mathcal{ELU}$-TBoxes is CONP-complete regarding data complexity while it is in PTIME for $\mathcal{EL}$-TBoxes and thus it is possible to achieve tractability by replacing a classical $\mathcal{ELU}$-TBox with its $\mathcal{EL}$-approximation. Interestingly, such a replacement is *optimal* in the following sense: for a TBox $\mathcal{T}$, we say that *answering instance queries w.r.t.* $\mathcal{T}$ *is in* PTIME if for every instance query $C(a)$, there is a polytime algorithm that, given an ABox $\mathcal{A}$, decides whether $\mathcal{A}, \mathcal{T} \models C(a)$ (see (Lutz and Wolter 2012) for more information on this *non-uniform view* of data complexity); then, the $\mathcal{EL}$-approximation of an $\mathcal{ELU}$-TBox $\mathcal{T}$ is *the most specific TBox* among all general $\mathcal{ELU}$-TBoxes $\mathcal{T}'$ that are entailed by $\mathcal{T}$ such that answering instance queries w.r.t. $\mathcal{T}'$ is in PTIME. By the results in (Lutz and Wolter 2012), the same is true for the more general conjunctive queries.

Theorem 7. *Let $\mathcal{T}$ be a (general) $\mathcal{ELU}$-TBox and $\mathcal{T}'$ the $\mathcal{EL}$-approximant of $\mathcal{T}$. If $\mathcal{T}''$ is a (general) $\mathcal{ELU}$-TBox with $\mathcal{T} \models \mathcal{T}''$ and answering instance queries w.r.t. $\mathcal{T}''$ is in* PTIME*, then $\mathcal{T}' \models \mathcal{T}''$ (unless* PTIME=CONP*).*

To give a semantic characterization of $\mathcal{EL}$-approximants, we need the product operation on interpretations, as used in an $\mathcal{EL}$ context in (Lutz, Piro, and Wolter 2011). Given a family of interpretations $(\mathcal{I}_i)_{i \in I}$, the *product* $\mathcal{I} = \prod_{i \in I} \mathcal{I}_i$ is defined as follows: the domain $\Delta^\mathcal{I}$ consists of all functions $f : I \to \bigcup_{i \in I} \Delta^{\mathcal{I}_i}$ with $f(i) \in \Delta^{\mathcal{I}_i}$ for all $i \in I$ and all symbols $A \in \mathsf{N_C}$ and $r \in \mathsf{N_R}$ are interpreted as follows:

- $f \in A^\mathcal{I}$ iff $f(i) \in A^{\mathcal{I}_i}$, for all $i \in I$
- $(f, g) \in r^\mathcal{I}$ iff $(f(i), g(i)) \in r^{\mathcal{I}_i}$, for all $i \in I$.

The connection between $\mathcal{EL}$-concepts and products follows.

Lemma 8. *For all $\mathcal{EL}$-concepts C, families of interpretations $(\mathcal{I}_i)_{i \in I}$, and $f \in \Delta^{\Pi_{i \in I} \mathcal{I}_i}$, we have $f \in C^{\Pi_{i \in I} \mathcal{I}_i}$ iff $f(i) \in C^{\mathcal{I}_i}$ for all $i \in I$.*

Now for the semantic characterization of approximants. For a class $\mathfrak{C}$ of interpretations, we use $\mathsf{cl}_\Pi(\mathfrak{C})$ to denote the *closure under products* of $\mathfrak{C}$, i.e., the class of all interpretations $\mathcal{I}$ such that $\mathcal{I} = \prod_{i \in I} \mathcal{I}_i$ for some (potentially infinite) family of interpretations $(\mathcal{I}_i)_{i \in I}$ contained in $\mathfrak{C}$. We use $\mathfrak{C}_{\restriction \mathsf{finout}}$ to denote the restriction of $\mathfrak{C}$ to interpretations of finite outdegree. It was shown in (Lutz, Piro, and Wolter 2011) that for each $\mathcal{EL}$-TBox $\mathcal{T}$, $\mathsf{mod}(\mathcal{T})$ is closed under products.

Theorem 9. *Let $\mathcal{T}$ be an FO-TBox and $\mathcal{T}'$ an $\mathcal{EL}$-TBox with $\mathcal{T} \models \mathcal{T}'$. Then $\mathcal{T}'$ is an $\mathcal{EL}$-approximant of $\mathcal{T}$ iff* $\mathsf{mod}(\mathcal{T}')_{\restriction \mathsf{finout}} = \mathsf{cl}_\approx(\mathsf{cl}_\Pi(\mathsf{mod}(\mathcal{T})))_{\restriction \mathsf{finout}}$.

The central ingredient to the proof of Theorem 9 is the following result.

Lemma 10. *Let $\mathcal{T}$ be an $\mathcal{EL}$-TBox. The following are equivalent for any interpretation $\mathcal{I}$ of finite outdegree:*

1. *$\mathcal{I} \models C \sqsubseteq D$, for all $\mathcal{EL}$-CIs $C \sqsubseteq D$ with $\mathcal{T} \models C \sqsubseteq D$;*
2. *$\mathcal{I} \in \mathsf{cl}_\approx(\mathsf{cl}_\Pi(\mathsf{mod}(\mathcal{T})))$.*

Proof. We prove that Point 2 implies Point 1 and defer the other direction, which is more involved, to the long version. Assume that $\mathcal{I} \in \mathsf{cl}_\approx(\mathsf{cl}_\Pi(\mathsf{mod}(\mathcal{T})))$, but $d \in C^\mathcal{I} \setminus D^\mathcal{I}$ for some $d \in \Delta^\mathcal{I}$ and $\mathcal{EL}$-CI $C \sqsubseteq D$ with $\mathcal{T} \models C \sqsubseteq D$. Take a family $(\mathcal{I}_i)_{i \in I}$ of models of $\mathcal{T}$ and an $f \in \Delta^{\prod_{i \in I} \mathcal{I}_i}$ such that $(\mathcal{I}, d) \approx (\prod_{i \in I} \mathcal{I}_i, f)$. By Lemma 4, $f \in C^{\prod_{i \in I} \mathcal{I}_i} \setminus D^{\prod_{i \in I} \mathcal{I}_i}$ and by Lemma 8, there is an $i \in I$ with $f(i) \in C^{\mathcal{I}_i} \setminus D^{\mathcal{I}_i}$ in contradiction to $\mathcal{I}_i$ being a model of $\mathcal{T}$. ❑

To show that Theorem 9 is a consequence of Lemma 10, one uses the well-known fact that whenever $\mathcal{T}' \not\models C \sqsubseteq D$ for an $\mathcal{EL}$-TBox $\mathcal{T}'$, then there is an $\mathcal{I} \in \mathsf{mod}(\mathcal{T}')_{\restriction \mathsf{finout}}$ with $C^\mathcal{I} \setminus D^\mathcal{I} \neq \emptyset$. It remains open whether Theorem 9 holds without the restriction to finite outdegree.

5 $\mathcal{EL}$ automata

We introduce $\mathcal{EL}$ automata as a novel automaton model and establish some central properties. Like the automata introduced in (Janin and Walukiewicz 1995; Wilke 2001), an $\mathcal{EL}$ automaton $\mathcal{A}$ runs directly on interpretations. As we will see, $\mathcal{EL}$ automata share many crucial properties with $\mathcal{EL}$-TBoxes. Yet, they are strictly more expressive and we will prove that, in a sense to be made precise later, uniform $\mathcal{EL}_\Sigma$-interpolants of $\mathcal{EL}$-TBoxes and $\mathcal{EL}$-approximants of classical $\mathcal{ELU}$-TBoxes can *always* be represented by a (finite!) $\mathcal{EL}$ automaton. As illustrated by the examples in Sections 3 and 4, this is not the case for $\mathcal{EL}$-TBoxes.

Definition 11. An $\mathcal{EL}$ *automaton (EA)* is a tuple $\mathcal{A} = (Q, P, \Sigma_N, \Sigma_E, \delta)$, where Q is a finite set of *bottom up states*, P is a finite set of *top down states*, $\Sigma_N \subseteq \mathsf{N_C}$ is the finite *node alphabet*, $\Sigma_E \subseteq \mathsf{N_R}$ is the finite *edge alphabet*, and δ is a set of transitions of the following form:

$$\begin{array}{rclcrcl} \mathsf{true} & \to & q & \quad & p & \to & p_1 \\ A & \to & q & & p & \to & \langle r \rangle p_1 \\ q_1 \wedge \cdots \wedge q_n & \to & q & & p & \to & A \\ \langle r \rangle q_1 & \to & q & & p & \to & \mathsf{false} \\ q & \to & p & & & & \end{array}$$

where q, $q_1, \ldots, q_n$ range over Q, p, p_1 range over P, A ranges over Σ_N, and r ranges over Σ_E.

The separation of states into bottom up states and top down states is crucial for attaining some relevant properties of $\mathcal{EL}$ automata, as discussed in more detail below.

Definition 12. Let $\mathcal{I}$ be an interpretation and $\mathcal{A} = (Q, P, \Sigma_N, \Sigma_E, \delta)$ an EA. A *run* of $\mathcal{A}$ on $\mathcal{I}$ is a map $\rho : \Delta^\mathcal{I} \to 2^{Q \cup P}$ such that for all $d \in \Delta^\mathcal{I}$, we have:

1. if $\mathsf{true} \to q \in \delta$, then $q \in \rho(d)$;
2. if $A \to q \in \delta$ and $d \in A^\mathcal{I}$, then $q \in \rho(d)$;
3. if $q_1, \ldots, q_n \in \rho(d)$ and $q_1 \wedge \cdots \wedge q_n \to q \in \delta$, then $q \in \rho(d)$;

4. if $(d,e) \in r^\mathcal{I}$, $q_1 \in \rho(e)$, and $\langle r\rangle q_1 \to q \in \delta$, then $q \in \rho(d)$;
5. if $q \in \rho(d)$ and $q \to p \in \delta$, then $p \in \rho(d)$;
6. if $p \in \rho(d)$ and $p \to p_1 \in \delta$, then $p_1 \in \rho(d)$;
7. if $p \in \rho(d)$ and $p \to \langle r\rangle p_1 \in \delta$, then there is an $(d,e) \in r^\mathcal{I}$ with $p_1 \in \rho(e)$;
8. if $p \in \rho(d)$ and $p \to A \in \delta$, then $d \in A^\mathcal{I}$;
9. if $p \to \mathsf{false} \in \delta$, then $p \notin \rho(d)$.

We use $L(\mathcal{A})$ to denote the language accepted by $\mathcal{A}$, i.e., the set of interpretations $\mathcal{I}$ such that there is a run of $\mathcal{A}$ on $\mathcal{I}$.

Note that EAs run on interpretations that interpret all symbols in $\mathsf{N_C} \cup \mathsf{N_R}$, not just those in $\Sigma_N \cup \Sigma_E$. However, the transition relation can only use symbols from the latter set. Thus, the alphabets of an EA play the same role as a signature of an $\mathcal{EL}$-TBox.

As an example, consider the following EA, which accepts precisely those interpretations that satisfy the (non-$\mathcal{EL}$) CI $A_1 \sqcap \exists r^*.A_2 \sqsubseteq B$, where r^* is interpreted as the transitive and reflexive closure of r (a property not expressible by an $\mathcal{EL}$-TBox):

$$
\begin{array}{rclcrcl}
Q &=& \{q_{A_1}, q_{A_2}, q_\wedge\} & & P &=& \{p_B\} \\
\Sigma_N &=& \{A, B\} & & \Sigma_E &=& \{r, s\} \\
\delta &=& \{ \quad A_1 \;\to\; q_{A_1} & & A_2 &\to& q_{A_2} \\
& & \langle r\rangle q_{A_2} \;\to\; q_{A_2} & & q_{A_1} \wedge q_{A_2} &\to& q_\wedge \\
& & q_\wedge \to p_B & & p_B &\to& B \quad \}.
\end{array}
$$

The above example shows that EAs are more expressive than $\mathcal{EL}$-TBoxes. Conversely, every $\mathcal{EL}$-TBox is equivalent to some EA. We say that an EA $\mathcal{A}$ is *equivalent* to a TBox $\mathcal{T}$ if $L(\mathcal{A}) = \mathsf{mod}(\mathcal{T})$.

Proposition 13. *Every $\mathcal{EL}$-TBox $\mathcal{T}$ can be converted in polynomial time into an equivalent EA $\mathcal{A}_\mathcal{T}$.*

To prove Proposition 13, let $\mathcal{T}$ be a TBox, $\mathsf{sub}(\mathcal{T})$ the subconcepts of (concepts that occur in) $\mathcal{T}$, and define

$$\mathcal{A}_\mathcal{T} = (Q, P, \mathsf{sig}(\mathcal{T}) \cap \mathsf{N_C}, \mathsf{sig}(\mathcal{T}) \cap \mathsf{N_R}, \delta)$$

where

- $Q = \{q_C \mid C \in \mathsf{sub}(\mathcal{T})\}$, $P = \{p_C \mid C \in \mathsf{sub}(\mathcal{T})\}$;
- δ consists of the following transitions:
  - $\mathsf{true} \to q_\top$ if $\top \in \mathsf{sub}(\mathcal{T})$;
  - $A \to q_A$ for all $A \in \mathsf{sig}(\mathcal{T}) \cap \mathsf{N_C}$;
  - $q_C \wedge q_D \to q_{C\sqcap D}$;
  - $\langle r\rangle q_C \to q_{\exists r.C}$ for all $\exists r.C \in \mathsf{sub}(\mathcal{T})$;
  - $q_C \to p_D$ for all $C, D \in \mathsf{sub}(\mathcal{T})$ with $\mathcal{T} \models C \sqsubseteq D$;
  - $p_A \to A$ for all $A \in \mathsf{sig}(\mathcal{T}) \cap \mathsf{N_C}$;
  - $p_{\exists r.C} \to \langle r\rangle p_C$ for all $\exists r.C \in \mathsf{sub}(\mathcal{T})$;
  - $p_C \to p_D$ for all $C, D \in \mathsf{sub}(\mathcal{T})$ with $\mathcal{T} \models C \sqsubseteq D$;
  - $p_\bot \to \mathsf{false}$ if $\bot \in \mathsf{sub}(\mathcal{T})$.

Note that $\mathcal{A}_\mathcal{T}$ can be constructed in polynomial time since $\mathcal{EL}$-subsumptions $\mathcal{T} \models C \sqsubseteq D$ can be decided in PTIME (Baader, Brandt, and Lutz 2005).

Lemma 14. $L(\mathcal{A}_\mathcal{T}) = \mathsf{mod}(\mathcal{T})$.

Proof.(sketch) "$\supseteq$". Let $\mathcal{I} \in \mathsf{mod}(\mathcal{T})$. Define $\rho : \Delta^\mathcal{I} \to 2^{P\cup Q}$ by setting for $d \in \Delta^\mathcal{I}$

$$\rho(d) = \{q_C, p_C \mid d \in C^\mathcal{I}, C \in \mathsf{sub}(\mathcal{T})\}$$

One can show that ρ is a run of $\mathcal{A}_\mathcal{T}$ on $\mathcal{I}$. Hence $\mathcal{I} \in L(\mathcal{A}_\mathcal{T})$ as required.

"$\subseteq$". Let $\mathcal{I} \in L(\mathcal{A}_\mathcal{T})$. It can be proved by induction on the structure of C that for all $\mathcal{EL}$-concepts $C \in \mathsf{sub}(\mathcal{T})$, all runs ρ and all $d \in \Delta^\mathcal{I}$, we have

1. $d \in C^\mathcal{I}$ implies $q_C \in \rho(d)$;
2. $p_C \in \rho(d)$ implies $d \in C^\mathcal{I}$.

Since $\mathcal{A}_\mathcal{T}$ has transitions $q_C \to q_D$ and $q_D \to p_D$ for each $C \sqsubseteq D \in \mathcal{T}$, it follows that $\mathcal{I}$ satisfies $\mathcal{T}$. ❑

We now analyze some basic properties of $\mathcal{EL}$ automata, in particular the complexity of deciding emptiness and containment as well as closure properties of languages accepted by a single EA and the class of languages accepted by EAs in general.

Theorem 15. *For EAs, emptiness can be decided in* PTIME *and containment is* EXPTIME*-complete.*

Proof. (sketch) The PTIME upper bound for EA emptiness is proved by a straightforward reduction to unsatisfiability of $\mathcal{EL}$ concepts w.r.t. general TBoxes (recall that, in this paper, $\mathcal{EL}$ includes $\bot$), which is in PTIME (Baader, Brandt, and Lutz 2005); essentially, states are translated into concept names and each transition gives rise to one CI. The lower bound for containment is a consequence of the fact that deciding $\mathcal{EL}$-conservative extensions is EXPTIME-hard (Lutz and Wolter 2010) and, by Theorem 28 proved later on, can be reduced in polynomial time to containment of EAs. The upper bound is obtained by a polytime translation of EAs into alternating parity tree automata in the style of Wilke for which containment is known to be in EXPTIME (Wilke 2001). ❑

We now turn to closure properties. An interpretation $\mathcal{I}$ is the *disjoint union* of a (potentially infinite) family of interpretations $(\mathcal{I}_i)_{i\in I}$ with pairwise disjoint domains if $\Delta^\mathcal{I} = \bigcup_{i\in I} \Delta^{\mathcal{I}_i}$ and $X^\mathcal{I} = \bigcup_{i\in I} X^{\mathcal{I}_i}$ for all $X \in \mathsf{N_C} \cup \mathsf{N_R}$. For a class $\mathfrak{C}$ of interpretations, we use $\mathsf{cl}_\Delta(\mathfrak{C})$ to denote the *closure under disjoint unions* of $\mathfrak{C}$, i.e., the class of all interpretations $\mathcal{I}$ such that $\mathcal{I}$ is the disjoint union of a family of interpretations $(\mathcal{I}_i)_{i\in I}$ contained in $\mathfrak{C}$. It was shown in (Lutz, Piro, and Wolter 2011) that an FO-TBox $\mathcal{T}$ is equivalent to an $\mathcal{EL}$-TBox if and only if $\mathsf{mod}(\mathcal{T})$ is closed under global equisimulations, products, and disjoint unions. We show that EAs enjoy the same closure properties as $\mathcal{EL}$-TBoxes.

Lemma 16. *For every EA $\mathcal{A}$, $L(\mathcal{A})$ is closed under global equisimulations, products, and disjoint unions.*

Proof. $\mathsf{cl}_\Delta(L(\mathcal{A})) \subseteq L(\mathcal{A})$. Let $\mathcal{I}$ be the disjoint union of $\mathcal{I}_i \in L(\mathcal{A})$, $i \in I$, and let ρ_i be a run of $\mathcal{A}$ on $\mathcal{I}_i$. Then $\rho = \bigcup_{i\in I} \rho_i$ is a run of $\mathcal{A}$ on $\mathcal{I}$. Hence $\mathcal{I} \in L(\mathcal{A})$.

$\mathsf{cl}_\Pi(L(\mathcal{A})) \subseteq L(\mathcal{A})$. Let $\mathcal{I}$ be the product of $\mathcal{I}_i \in L(\mathcal{A})$, $i \in I$, and let ρ_i be a run of $\mathcal{A}$ on $\mathcal{I}_i$. Define $\rho : \prod_{i\in I} \Delta^{\mathcal{I}_i} \to 2^{Q\cup P}$ by setting for $f \in \prod_{i\in I} \Delta^{\mathcal{I}_i}$: $\rho(f) = \bigcap_{i\in I} \rho(f(i))$ One can show that ρ is a run of $\mathcal{A}$ on $\mathcal{I}$. Hence $\mathcal{I} \in L(\mathcal{A})$.

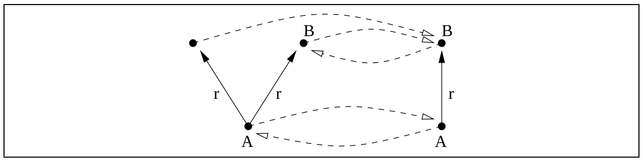

Figure 1: $\mathcal{I}_1$ and $\mathcal{I}_2$ with mutual simulations

$\mathsf{cl}_{\approx}(L(\mathcal{A})) \subseteq L(\mathcal{A})$. Let $\mathcal{I} \in \mathsf{cl}_{\approx}(L(\mathcal{A}))$. Take for every $d \in \Delta^{\mathcal{I}}$ an $\mathcal{I}_d \in L(\mathcal{A})$ and $f_d \in \Delta^{\mathcal{I}_d}$ such that $(\mathcal{I}, d) \approx (\mathcal{I}_d, f_d)$. Let $\mathcal{J}$ be the disjoint union of the $\mathcal{I}_d$, $d \in \Delta^{\mathcal{I}}$. Then $\mathcal{J} \in L(\mathcal{A})$, by Point 1. Let ρ_1 be a run of $\mathcal{A}$ on $\mathcal{J}$. Define ρ_2 by setting for $d \in \Delta^{\mathcal{I}_2}$:

$$\rho_2(d) = (Q \cap \bigcap_{(\mathcal{I},d) \leq (\mathcal{J},d')} \rho_1(d')) \cup (P \cap \bigcup_{(\mathcal{I},d) \geq (\mathcal{J},d')} \rho_1(d'))$$

One can show that ρ_2 is a run of $\mathcal{A}$ on $\mathcal{I}$. ❑

We conjecture that the operations of Lemma 16 characterize languages $L(\mathcal{A})$ of EAs $\mathcal{A}$ within MSO (monadic second-order logic). An MSO-TBox is defined in the same way as an FO-TBox with the exception that we admit second-order quantifiers for sets.

Conjecture 17. *For any MSO-TBox $\mathcal{T}$, there exists an EA $\mathcal{A}$ with $\mathsf{mod}(\mathcal{T}) = L(\mathcal{A})$ iff $\mathsf{mod}(\mathcal{T})$ is closed under global equisimulations, products, and disjoint unions.*

Note that closure under global equisimulations fails when we give up the separation of states into bottom-up and top-down states. To see this, consider the following non-EA:

$$\begin{array}{rclcrcl} Q & = & \{q, q'\} & & P & = & \{p_B\} \\ \Sigma_N & = & \{A, B\} & & \Sigma_E & = & \{r\} \\ \delta & = & \multicolumn{5}{l}{\{A \rightarrow \langle r \rangle q,\ B \rightarrow q',\ q \wedge q' \rightarrow \bot\ \}.} \end{array}$$

With the natural definition of runs, this EA would accept the interpretation $\mathcal{I}_1$ on the left-hand side of Figure 1, but not the interpretation $\mathcal{I}_2$ on the right-hand side. However, as indicated by the dashed edges, $\mathcal{I}_2 \in \mathsf{cl}_{\approx}(\{\mathcal{I}_1\})$.

In passing, we also summarize some closure properties of the class of languages that can be accepted with the new automaton model.

Observation 18. *The class of languages accepted by EAs is closed under intersection. It is not closed under complementation and union.*

6 From $\mathcal{EL}$ automata to $\mathcal{EL}$-TBoxes

We start with making precise what it means for an $\mathcal{EL}$ automaton to represent a uniform interpolant or an $\mathcal{EL}$-approximant. For an EA $\mathcal{A}$, we write $\mathcal{A} \models C \sqsubseteq D$ if every $\mathcal{I} \in L(\mathcal{A})$ satisfies $C \sqsubseteq D$. This allows us to speak about $\mathcal{EL}_\Sigma$-entailment and $\mathcal{EL}_\Sigma$-inseparability between EAs and TBoxes. For example, an EA $\mathcal{A}$ and a TBox $\mathcal{T}$ are $\mathcal{EL}_\Sigma$-inseparable, in symbols $\mathcal{A} \equiv_\Sigma^{\mathcal{EL}} \mathcal{T}$, if $\mathcal{A} \models C \sqsubseteq D$ iff $\mathcal{T} \models C \sqsubseteq D$ for all $\mathcal{EL}_\Sigma$ inclusions $C \sqsubseteq D$. We say that $\mathcal{A}$

- *represents the uniform $\mathcal{EL}_\Sigma$-interpolant of an $\mathcal{EL}$-TBox $\mathcal{T}$* if $\mathcal{A} \equiv_\Sigma^{\mathcal{EL}} \mathcal{T}$ and $\Sigma = \Sigma_N \cup \Sigma_E$ is the signature of $\mathcal{A}$ (see the note after Definition 1);
- *represents the $\mathcal{EL}$-approximant of a classical $\mathcal{ELU}$-TBox $\mathcal{T}$* if $\mathcal{A} \equiv^{\mathcal{EL}} \mathcal{T}$ (see the note after Definition 6).

A main aim of this paper is to give decision procedures for the existence of uniform interpolants and $\mathcal{EL}$-approximants (represented as a TBox). The approach is to construct an EA $\mathcal{A}$ that represents the interpolant/approximant, and then to decide for a suitable signature Σ whether there exists an $\mathcal{EL}_\Sigma$-TBox $\mathcal{T}$ such that $\mathcal{A} \equiv_\Sigma^{\mathcal{EL}} \mathcal{T}$. In this section, we show that the latter can be done in ExpTime. In fact, we will later construct the relevant EAs $\mathcal{A}$ such that the 'suitable signature' Σ is simply the signature of $\mathcal{A}$. Our aim is thus to prove the following.

Theorem 19. *Given an EA $\mathcal{A} = (Q, P, \Sigma_N, \Sigma_E, \delta)$, it can be decided in ExpTime whether there is an $\mathcal{EL}_\Sigma$-TBox $\mathcal{T}$ such that $\mathcal{A} \equiv_\Sigma^{\mathcal{EL}} \mathcal{T}$, where $\Sigma = \Sigma_N \cup \Sigma_E$.*

To prove Theorem 19, we proceed in two steps: first, we characterize the non-existence of an $\mathcal{EL}_\Sigma$-TBox $\mathcal{T}$ with $\mathcal{A} \equiv_\Sigma^{\mathcal{EL}} \mathcal{T}$ in terms of the existence of certain interpretations, and then we show that the latter can be checked in ExpTime using alternating parity tree automata (APTAs) in the form defined by Wilke (Wilke 2001). The technical tools developed in this section can be regarded as '$\mathcal{EL}$-analogues' of the tools developed for deciding the existence of uniform $\mathcal{ALC}$-interpolants in (Lutz and Wolter 2011). All proofs are deferred to the long version of this paper.

Let $\mathcal{A} = (Q, P, \Sigma_N, \Sigma_E, \delta)$ be an EA and $\Sigma = \Sigma_N \cup \Sigma_E$. First observe that it is trivial to find an *infinite* TBox that is $\mathcal{EL}_\Sigma$-inseparable from $\mathcal{A}$, namely the set

$$\mathcal{T}_\Sigma(\mathcal{A}) = \{C \sqsubseteq D \mid \mathcal{A} \models C \sqsubseteq D,\ C, D\ \mathcal{EL}_\Sigma\text{-concepts}\}.$$

Candidates for finite TBoxes with the same property are provided by the following subsets of $\mathcal{T}_\Sigma(\mathcal{A})$. The *role depth* $\mathsf{rd}(C)$ of a concept C is the nesting depth of existential restrictions in C. For every $m \geq 0$, we can fix a *finite* set $\mathcal{EL}_f^m(\Sigma)$ of $\mathcal{EL}_\Sigma$-concepts D with $\mathsf{rd}(D) \leq m$ such that every $\mathcal{EL}_\Sigma$-concept C with $\mathsf{rd}(C) \leq m$ is equivalent to some $D \in \mathcal{EL}_f^m(\Sigma)$. Set

$$\mathcal{T}_\Sigma^m(\mathcal{A}) = \{C \sqsubseteq D \mid \mathcal{A} \models C \sqsubseteq D \text{ and } C, D \in \mathcal{EL}_f^m(\Sigma)\}.$$

Obviously, the following are equivalent:

(a) no (finite!) $\mathcal{EL}_\Sigma$-TBox is $\mathcal{EL}_\Sigma$-inseparable from $\mathcal{A}$;

(b) $\mathcal{T}_\Sigma^m(\mathcal{A})$ is not $\mathcal{EL}_\Sigma$-inseparable from $\mathcal{A}$, for all $m \geq 0$;

(c) for all $m \geq 0$ there is a $k > m$ s.t. $\mathcal{T}_\Sigma^m(\mathcal{A}) \not\models \mathcal{T}_\Sigma^k(\mathcal{A})$.

We now present the announced characterization of (a). An interpretation $\mathcal{I}$ is a *tree-interpretation* if $(\Delta^{\mathcal{I}}, \bigcup_{r \in \mathsf{N_R}} r^{\mathcal{I}})$ is a (possibly infinite) tree and $r^{\mathcal{I}} \cap s^{\mathcal{I}} = \emptyset$ for any two distinct $r, s \in \mathsf{N_R}$. By $\rho^{\mathcal{I}}$ we denote the root of $\mathcal{I}$. Note that every $\mathcal{EL}$-TBox $\mathcal{T}$ is determined by tree interpretations: if $\mathcal{T} \not\models C \sqsubseteq D$, then there exists a tree interpretation $\mathcal{I}$ that is a model of $\mathcal{T}$ such that $\rho^{\mathcal{I}} \in C^{\mathcal{I}} \setminus D^{\mathcal{I}}$. For $m \geq 0$, we use $\mathcal{I}^{\leq m}$ to denote the restriction of $\mathcal{I}$ to those elements of $\Delta^{\mathcal{I}}$ that can be reached from $\rho^{\mathcal{I}}$ in at most m steps in the graph $(\Delta^{\mathcal{I}}, \bigcup_{r \in \mathsf{N_R}} r^{\mathcal{I}})$. For any $d \in \Delta^{\mathcal{I}}$, $\mathcal{I}(d)$ denotes the restriction of $\mathcal{I}$ to those elements of $\Delta^{\mathcal{I}}$ that can be reached from d in the graph $(\Delta^{\mathcal{I}}, \bigcup_{r \in \mathsf{N_R}} r^{\mathcal{I}})$.

The proof of the following result consists of an adaptation of the proof of Theorem 9 in (Lutz and Wolter 2011) and additionally requires the use and analysis of "m-equisimulations", which capture the expressive power of $\mathcal{EL}$-concept of role-depth $\leq m$.

Theorem 20. *Let $\mathcal{A}$ be an EA. For all $m > 0$, the following conditions are equivalent:*

1. *There exists $k > m$ such that $\mathcal{T}_\Sigma^m(\mathcal{A}) \not\models \mathcal{T}_\Sigma^k(\mathcal{A})$;*
2. *there exist two tree interpretations, $\mathcal{I}_1$ and $\mathcal{I}_2$, of finite outdegree such that*
   - *$\mathcal{I}_1^{\leq m} = \mathcal{I}_2^{\leq m}$;*
   - *$\mathcal{I}_1 \in L(\mathcal{A})$;*
   - *$\mathcal{I}_2 \notin L(\mathcal{A})$;*
   - *$\mathcal{I}_2(d) \in L(\mathcal{A})$, for all d such that $(\rho^{\mathcal{I}_2}, d) \in r^{\mathcal{I}_2}$ for some r.*

To provide the announced characterization of (a), we prove that rather than testing Point 2 of Theorem 20 for all m, it suffices to consider a single number m. Technically, this is achieved by a pumping argument as in (Lutz and Wolter 2011). Since we are concerned with $\mathcal{EL}$ rather than with $\mathcal{ALC}$, we obtain a single exponential bound on m rather than a double exponential one. The proof makes intensive use of what we call *canonical pre-runs* of EAs, the automaton counterpart of the well-known canonical models of an $\mathcal{EL}$-TBox (Lutz, Toman, and Wolter 2009).

Theorem 21. *Let $\mathcal{A} = (Q, P, \Sigma_N, \Sigma_E, \delta)$ be an EA and $\Sigma = \Sigma_N \cup \Sigma_E$. There does not exist an $\mathcal{EL}_\Sigma$-TBox $\mathcal{T}$ with $\mathcal{A} \equiv_\Sigma^{\mathcal{EL}} \mathcal{T}$ iff Point 2 of Theorem 20 holds with $m = 2^{2|Q \cup P|} + 1$.*

We now turn to the second step in the proof of Theorem 19 and show that the characterization provided by Theorem 21 leads to an EXPTIME decision procedure with an APTA emptiness check at its basis. We refer to the long version of this paper for a precise definition of APTAs. Like EAs, APTAs run directly on interpretations (actually, pointed interpretations). The proof of the following result is an easy adaptation of the proof of the corresponding theorem in (Lutz and Wolter 2011).

Theorem 22. *Let $\mathcal{A} = (Q, P, \Sigma_N, \Sigma_E, \delta)$ be an EA, $\Sigma = \Sigma_N \cup \Sigma_E$, and $m \geq 0$. Then there is an APTA $\mathcal{A}_{\mathcal{A},\Sigma,m}$ with state set Q and node and edge alphabets Σ'_N and Σ'_E such that $L(\mathcal{A}_{\mathcal{A},\Sigma,m}) \neq \emptyset$ iff Point 2 of Theorem 20 is satisfied. Moreover, $|Q| \in \mathcal{O}(n + \log^2 m)$ and $|\Sigma'_N|, |\Sigma'_E| \in \mathcal{O}(n + \log m)$, where $n = |Q \cup P|$.*

The size of $\mathcal{A}_{\mathcal{T},\Sigma,m}$ is polynomial in $|Q \cup P|$ and logarithmic in m. By Theorem 21, we can use $m = 2^{2|Q \cup P|}+1$, and thus the size of $\mathcal{A}_{\mathcal{T},\Sigma,m}$ is polynomial in $|Q \cup P|$. As emptiness of APTAs can be decided in EXPTIME, we obtain an EXPTIME decision procedure for (a).

The above algorithm decides for a given EA $\mathcal{A}$ whether there is an $\mathcal{EL}_\Sigma$-TBox that has the same $\mathcal{EL}_\Sigma$-CIs as consequences as $\mathcal{A}$, with Σ the signature of $\mathcal{A}$. It is interesting to note (and will be useful in the subsequent section) that this is equivalent to the existence of an $\mathcal{EL}_\Sigma$-TBox $\mathcal{T}$ that is *equivalent to* $\mathcal{A}$ in the sense that $L(\mathcal{A}) = \mathsf{mod}(\mathcal{T})$. This is a consequence of Proposition 13 and the following, proved in the appendix.

Proposition 23. *For all EAs $\mathcal{A}_1, \mathcal{A}_2$ over the same alphabets Σ_N and Σ_E, we have $\mathcal{T}_\Sigma(\mathcal{A}_2) \subseteq \mathcal{T}_\Sigma(\mathcal{A}_1)$ iff $L(\mathcal{A}_1) \subseteq L(\mathcal{A}_2)$ when $\Sigma = \Sigma_N \cup \Sigma_E$.*

7 Automata for Uniform Interpolants and Conservative Extensions

We prove that deciding the existence of uniform $\mathcal{EL}_\Sigma$-interpolants is EXPTIME-complete. We also show that, using EAs and bypassing the machinery in Section 6, conservative extensions can be decided in EXPTIME. A matching lower bound is known from (Lutz and Wolter 2010). We first show that for every $\mathcal{EL}$-TBox $\mathcal{T}$ and signature Σ, there is an EA that represents the uniform $\mathcal{EL}_\Sigma$-interpolant of $\mathcal{T}$. In fact, the EA is even stronger than required at the beginning of Section 6 as it accepts precisely the 'right' class of models rather than only have the 'right' $\mathcal{EL}$-consequences.

Theorem 24. *Let $\mathcal{T}$ be an $\mathcal{EL}$-TBox and $\Sigma \subseteq \mathsf{sig}(\mathcal{T})$ a signature. Then one can construct in polynomial time an EA $\mathcal{A}_{\mathcal{T},\Sigma} = (Q, P, \Sigma \cap \mathsf{N_C}, \Sigma \cap \mathsf{N_R}, \delta)$ with $|Q \cup P| \in \mathcal{O}(|\mathcal{T}|)$ such that $L(\mathcal{A}_{\mathcal{T},\Sigma}) = \mathsf{cl}_\approx^\Sigma(\mathsf{mod}(\mathcal{T}))$.*

More specifically, the automaton $\mathcal{A}_{\mathcal{T},\Sigma}$ for Theorem 24 can be constructed as follows. Let $\mathcal{A}_\mathcal{T} = (Q, P, \Sigma_N, \Sigma_E, \delta)$ be the automaton from the proof of Proposition 13. We have $\Sigma \subseteq \Sigma_N \cup \Sigma_E$. The Σ-*restriction* of $\mathcal{A}_\mathcal{T}$ is the automaton $(Q, P, \Sigma \cap \Sigma_N, \Sigma \cap \Sigma_E, \delta')$ where δ' is obtained from δ by dropping all transitions $A \rightarrow q$ and $p \rightarrow A$ with $A \notin \Sigma$ and all transitions $\langle r \rangle q_1 \rightarrow q$ and $p \rightarrow \langle r \rangle p_1$ with $r \notin \Sigma$. Call the resulting automaton $\mathcal{A}_{\mathcal{T},\Sigma}$. By the following lemma, it is as required for Theorem 24.

Lemma 25. $L(\mathcal{A}_{\mathcal{T},\Sigma}) = \mathsf{cl}_\approx^\Sigma(\mathsf{mod}(\mathcal{T}))$.

Proof.(sketch) "$\supseteq$". By Lemma 16, it is sufficient to show that $\mathsf{mod}(\mathcal{T}) \subseteq L(\mathcal{A}_{\mathcal{T},\Sigma})$. This can be done as in the proof of Lemma 14.

"$\subseteq$". Let $\mathcal{I} \in L(\mathcal{A}_{\mathcal{T},\Sigma})$ and ρ be a run of $\mathcal{A}_{\mathcal{T},\Sigma}$ on $\mathcal{I}$. We construct a model $\mathcal{J}$ of $\mathcal{T}$ such that for all $d \in \Delta^\mathcal{I}$, $(\mathcal{I}, d) \approx_\Sigma (\mathcal{J}, d)$. Fix, for each $D = \exists r.C \in \mathsf{sub}(\mathcal{T})$ with $r \notin \Sigma$ and each $d \in \Delta^\mathcal{I}$ with $p_D \in \rho(d)$ a least tree model $\mathcal{J}_{C,d}$ of C and $\mathcal{T}$, i.e., $\mathcal{J}_{C,d}$ is a tree model of $\mathcal{T}$ with root $d \in C^{\mathcal{J}_{C,d}}$ and for all models $\mathcal{J}$ of $\mathcal{T}$ and all $e \in C^\mathcal{J}$, we have $(\mathcal{J}_{C,d}, d) \leq (\mathcal{J}, e)$. Moreover, let $\mathcal{J}_{D,d}$ be obtained from $\mathcal{J}_{C,d}$ by adding d as a fresh root which has an r-edge into the root of $\mathcal{J}_{C,d}$.

Assume w.l.o.g. that the domains of all chosen models are pairwise disjoint, and that each model $\mathcal{J}_{D,d}$ shares with $\mathcal{I}$ only the domain element d. Let Γ be the set of all models $\mathcal{J}_{D,d}$ chosen. Now let $\mathcal{J}$ be the (non-disjoint) union of $\mathcal{I}$ and all interpretations in Γ. In the long version, we carefully analyze $\mathcal{J}$ to show that it is a model of $\mathcal{T}$ and, as intended, $(\mathcal{I}, d) \approx_\Sigma (\mathcal{J}, e)$ for $e = d$. ❏

We are now able to deliver our promise and prove the "$\Rightarrow$" direction of Point 1 of Theorem 5 without the restriction to interpretations of finite outdegree. To this end, assume that $\mathcal{T}$ Σ-entails $\mathcal{T}'$ and let $\mathcal{I}$ be a model of $\mathcal{T}$. By definition, $\mathcal{T}_\Sigma(\mathcal{A}_{\mathcal{T}',\Sigma}) \subseteq \mathcal{T}_\Sigma(\mathcal{A}_{\mathcal{T},\Sigma})$. Hence, by Proposition 23, $L(\mathcal{A}_{\mathcal{T},\Sigma}) \subseteq L(\mathcal{A}_{\mathcal{T}',\Sigma})$. Summing up, we thus have

$$\mathcal{I} \in \mathsf{mod}(\mathcal{T}) \subseteq L(\mathcal{A}_{\mathcal{T},\Sigma}) \subseteq L(\mathcal{A}_{\mathcal{T}',\Sigma}) \subseteq \mathsf{cl}^\Sigma_\approx(\mathsf{mod}(\mathcal{T}')$$

and are done. We come to the main theorem of this section.

Theorem 26. *Given an $\mathcal{EL}$-TBox $\mathcal{T}$ and signature Σ, it can be decided in* ExpTime *whether there is an $\mathcal{EL}$-TBox that is the uniform $\mathcal{EL}_\Sigma$-interpolant of $\mathcal{T}$.*

Proof. By Theorem 24, $\mathcal{T}$ can be converted in polynomial time into an EA $\mathcal{A}_{\mathcal{T},\Sigma}$ such that $L(\mathcal{A}_{\mathcal{T},\Sigma}) = \mathsf{cl}^\Sigma_\approx(\mathsf{mod}(\mathcal{T}))$. It follows by Lemma 4 that $\mathcal{T}$ and $\mathcal{A}_{\mathcal{T},\Sigma}$ are $\mathcal{EL}_\Sigma$-inseparable, i.e., $\mathcal{A}_{\mathcal{T},\Sigma}$ represents the uniform $\mathcal{EL}_\Sigma$-interpolant of $\mathcal{T}$. It remains to apply Theorem 19. ❑

A matching lower bound can be proved by a careful analysis and adaptation of the ExpTime lower bound for deciding $\mathcal{EL}$-conservative extensions given in (Lutz and Wolter 2010).

Theorem 27. *Given an $\mathcal{EL}$-TBox $\mathcal{T}$ and signature Σ, it is* ExpTime*-hard to decide whether there is an $\mathcal{EL}$-TBox that is the uniform $\mathcal{EL}_\Sigma$-interpolant of $\mathcal{T}$.*

We now turn our attention to conservative extensions. It was originally proved in (Lutz and Wolter 2010) that deciding conservative extensions in $\mathcal{EL}$ is ExpTime-complete. We give an alternative proof of the upper bound based on EA containment, which is arguably more transparent than the original proof. Note that our proof completely bypasses the (somewhat intricate) machinery established in Section 6 and only relies on EA containment. The construction actually works for the more general problem of $\mathcal{EL}_\Sigma$-entailment.

Lemma 28. *Let $\mathcal{T}$ and $\mathcal{T}'$ be $\mathcal{EL}$-TBoxes and Σ a signature. Then $\mathcal{T}$ $\mathcal{EL}_\Sigma$-entails $\mathcal{T}'$ iff $L(\mathcal{A}_{\mathcal{T},\Sigma}) \subseteq L(\mathcal{A}_{\mathcal{T}',\Sigma})$.*

Proof. ($\Rightarrow$) Suppose $\mathcal{T}$ $\mathcal{EL}_\Sigma$-entails $\mathcal{T}'$ and $\mathcal{I} \in L(\mathcal{A}_{\mathcal{T},\Sigma})$. To prove that $\mathcal{I} \in L(\mathcal{A}_{\mathcal{T}',\Sigma})$, since $L(\mathcal{A}_{\mathcal{T}',\Sigma}) = \mathsf{cl}^\Sigma_\approx(\mathsf{mod}(\mathcal{T}'))$ it is sufficient to show $\mathcal{I} \in \mathsf{cl}^\Sigma_\approx(\mathsf{mod}(\mathcal{T}'))$. Let $d \in \Delta^\mathcal{I}$. Since $L(\mathcal{A}_{\mathcal{T},\Sigma}) = \mathsf{cl}^\Sigma_\approx(\mathsf{mod}(\mathcal{T}))$, there is a model $\mathcal{I}_1$ of $\mathcal{T}$ and a $d_1 \in \Delta^{\mathcal{I}_1}$ such that $(\mathcal{I}, d) \approx_\Sigma (\mathcal{I}_1, d_1)$. By Theorem 5, there is a model $\mathcal{I}_2$ of $\mathcal{T}'$ and a $d_2 \in \Delta^{\mathcal{I}_2}$ such that $(\mathcal{I}_1, d_1) \approx_\Sigma (\mathcal{I}_2, d_2)$. By closure under composition of Σ-equisimulations, $(\mathcal{I}, d) \approx_\Sigma (\mathcal{I}_2, d_2)$ as required.

($\Leftarrow$) Suppose $L(\mathcal{A}_{\mathcal{T},\Sigma}) \subseteq L(\mathcal{A}_{\mathcal{T}',\Sigma})$. By Theorem 5, it suffices to show that $\mathsf{mod}(\mathcal{T}) \subseteq \mathsf{cl}^\Sigma_\approx(\mathsf{mod}(\mathcal{T}'))$. To this aim, let $\mathcal{I}$ be a model of $\mathcal{T}$. Since $L(\mathcal{A}_{\mathcal{T},\Sigma}) = \mathsf{cl}^\Sigma_\approx(\mathsf{mod}(\mathcal{T})), \mathcal{I} \in L(\mathcal{A}_{\mathcal{T},\Sigma})$ and, therefore, $\mathcal{I} \in L(\mathcal{A}_{\mathcal{T}',\Sigma})$. Since $L(\mathcal{A}_{\mathcal{T}',\Sigma}) = \mathsf{cl}^\Sigma_\approx(\mathsf{mod}(\mathcal{T}'))$, this implies $\mathcal{I} \in \mathsf{cl}^\Sigma_\approx(\mathsf{mod}(\mathcal{T}'))$, as required. ❑

Together with Theorem 19, we obtain the desired result.

Theorem 29. *(Lutz and Wolter 2010) $\mathcal{EL}_\Sigma$-entailment and $\mathcal{EL}$-conservative extensions can be decided in* ExpTime.

8 Automata for Approximation

We prove that the existence of $\mathcal{EL}$-approximants of a classical $\mathcal{ELU}$-TBox can be decided in 2ExpTime using EAs. An ExpTime lower bound is established as well, but the precise complexity remains open.

In analogy to what was done in the previous section, the central observation is that for every classical $\mathcal{ELU}$-TBox $\mathcal{T}$, there is an EA $\mathcal{A}_\mathcal{T}$ of exponential size that represents the $\mathcal{EL}$-approximant of $\mathcal{T}$.

Theorem 30. *Let $\mathcal{T}$ be a classical $\mathcal{ELU}$-TBox. Then one can construct an EA $\mathcal{A}_\mathcal{T} = (Q, P, \mathsf{sig}(\mathcal{T}) \cap \mathsf{N_C}, \mathsf{sig}(\mathcal{T}) \cap \mathsf{N_R}, \delta)$ with $|Q \cup P| \in \mathcal{O}(2^{|\mathcal{T}|})$ such that $\mathcal{A}_\mathcal{T} \equiv^{\mathcal{EL}} \mathcal{T}$.*

More specifically, the automaton $\mathcal{A}_\mathcal{T}$ from Theorem 30 can be constructed as follows. Let $\mathcal{T}$ be a classical $\mathcal{ELU}$-TBox, $\mathsf{sub}(\mathcal{T})$ the subconcepts of $\mathcal{T}$, and $\mathsf{dis}(\mathcal{T})$ the set of all disjunctions of concepts from $\mathsf{sub}(\mathcal{T})$ that do not contain duplicate disjuncts. Define the EA $\mathcal{A}_\mathcal{T} = (Q, P, \mathsf{sig}(\mathcal{T}) \cap \mathsf{N_C}, \mathsf{sig}(\mathcal{T}) \cap \mathsf{N_R}, \delta)$ as follows:

- $Q = \{q_C \mid C \in \mathsf{dis}(\mathcal{T})\}$ and $P = \{p_C \mid C \in \mathsf{dis}(\mathcal{T})\}$;
- δ consists of the following transitions:
  - $\mathsf{true} \rightarrow q_\top$;
  - $A \rightarrow q_A$ and $q_A \rightarrow p_A$ for all $A \in \mathsf{sub}(C) \cap \mathsf{N_C}$;
  - $q_{C_1} \wedge \cdots \wedge q_{C_n} \rightarrow q_C$ for all $C_1, \ldots, C_n, C \in \mathsf{dis}(\mathcal{T})$ with $\mathcal{T} \models C_1 \sqcap \cdots \sqcap C_n \sqsubseteq C$;
  - $\langle r \rangle q_C \rightarrow q_D$ for all $C, D \in \mathsf{dis}(\mathcal{T})$ and $r \in \Sigma_E$ with $\mathcal{T} \models \exists r.C \sqsubseteq D$;
  - $q_C \rightarrow p_C$ for all $C \in \mathsf{dis}(\mathcal{T})$;
  - $p_C \rightarrow A$ for all $C \in \mathsf{dis}(\mathcal{T})$ and $A \in \mathsf{sub}(\mathcal{T}) \cap \mathsf{N_C}$ such that $\mathcal{T} \models C \sqsubseteq A$;
  - $p_C \rightarrow \langle r \rangle p_D$ for all $C, D \in \mathsf{dis}(\mathcal{T})$ and $r \in \Sigma_E$ such that $\mathcal{T} \models C \sqsubseteq \exists r.D$.

Using completeness w.r.t. interpretations of finite outdegree, Theorem 30 follows from the following lemma.

Lemma 31.

1. *If $\mathcal{I} \in L(\mathcal{A}_\mathcal{T})$, then $\mathcal{I} \models C \sqsubseteq D$ for all $\mathcal{EL}$-CIs $C \sqsubseteq D$ with $\mathcal{T} \models C \sqsubseteq D$.*
2. *If $\mathcal{I}$ has finite outdegree and $\mathcal{I} \models C \sqsubseteq D$ for all $\mathcal{EL}$-CIs with $\mathcal{T} \models C \sqsubseteq D$, then $\mathcal{I} \in L(\mathcal{A}_\mathcal{T})$.*

Proof. The proof of (1.) is given in the long version. For (2.), by Lemma 10, is is sufficient to show $\mathsf{cl}_\approx(\mathsf{cl}_\Pi(\mathsf{mod}(\mathcal{T}))) \subseteq L(\mathcal{A}_\mathcal{T})$. This follows from Lemma 16 if $\mathsf{mod}(\mathcal{T}) \subseteq L(\mathcal{A}_\mathcal{T})$. To show the inclusion let $\mathcal{I} \in \mathsf{mod}(\mathcal{T})$. Define a mapping $\rho : \Delta^\mathcal{I} \rightarrow 2^{P \cup Q}$ by setting $\rho(d) = \{q_C, p_C \mid d \in C^\mathcal{I}, C \in \mathsf{dis}(\mathcal{T})\}$. One can show that ρ is a run of $\mathcal{A}_\mathcal{T}$ on $\mathcal{I}$; hence $\mathcal{I} \in L(\mathcal{A}_\mathcal{T})$. ❑

Point 2 of Lemma 31 fails without the restriction to finite outdegree. To see this, consider the example TBox $\mathcal{T}$ about nodes in a tree given in Section 4. Let $\mathcal{I}$ be an interpretation that has a root node d_0 satisfying Node, Left, and Right and having outgoing succ-chains of unbounded finite length (but no infinite such chain!). $\mathcal{I}$ satisfies all $\mathcal{EL}$-CIs that follow from $\mathcal{T}$, but $\mathcal{I} \notin L(\mathcal{A}_\mathcal{T})$. We come to the main theorem of this section.

Theorem 32. *Given a classical $\mathcal{ELU}$-TBox $\mathcal{T}$, it can be decided in* 2ExpTime *whether there is an $\mathcal{EL}$-TBox that is the $\mathcal{EL}$-approximant of $\mathcal{T}$.*

Proof. Construct the automaton $\mathcal{A}_\mathcal{T}$ from the proof of Theorem 30 and apply the APTA-based decision procedure underlying Theorem 19. Despite the double exponential number of transitions of $\mathcal{A}_\mathcal{T}$, one can achieve a 2ExpTime-procedure by using on-the-fly versions of the construction of $\mathcal{A}_\mathcal{T}$ and of the APTA $\mathcal{A}_{\mathcal{A},\Sigma,m}$ from Theorem 22 when checking emptiness of the latter. ❏

We do not know whether the bound established in Theorem 32 is tight. However, we have the following.

Theorem 33. *Given a classical $\mathcal{ELU}$-TBox $\mathcal{T}$, it is* ExpTime*-hard to decide whether there is an $\mathcal{EL}$-TBox that is the $\mathcal{EL}$-approximant of $\mathcal{T}$.*

Proof. The proof is a modification of the ExpTime hardness proof in (Haase and Lutz 2008) for subsumption in classical $\mathcal{ELU}$-TBoxes. Denote by $\mathcal{EL}^\neg$ the extension of $\mathcal{EL}$ with negation. Since $\mathcal{EL}^\neg$ has the same expressive power as $\mathcal{ALC}$, it is ExpTime-complete to decide whether a classical $\mathcal{EL}^\neg$-TBox is satisfiable. We reduce this problem to deciding whether there is an $\mathcal{EL}$-approximant of an $\mathcal{ELU}$-TBox.

Let $\mathcal{T}$ be a classical $\mathcal{EL}^\neg$-TBox. We may assume w.l.o.g. that $\mathcal{T}$ contains only one role name r and is of the form

$$\{A_1 \equiv C_1, \ldots, A_n \equiv C_n\},$$

where each C_i is of the form $\top$, P, $\neg B$, $\exists r.B$ or $B_1 \sqcap B_2$ with P not occurring on the left-hand side of any CI in $\mathcal{T}$ (a 'primitive' concept name in $\mathcal{T}$) and B, B_1, B_2 occurring on a left-hand side ('defined' concept names in $\mathcal{T}$).

We convert $\mathcal{T}$ into a classical $\mathcal{ELU}$-TBox $\mathcal{T}'$ such that $\mathcal{T}$ is satisfiable iff there is no $\mathcal{EL}$-approximant of $\mathcal{T}'$. Introduce new concept names $\overline{A}_i$, $1 \leq i \leq n$, which intuitively represent $\neg A_i$. Let D be an abbreviation for

$$\bigsqcap_{1 \leq i \leq n} (A_i \sqcup \overline{A}_i).$$

We also use new concept names E_1, E_2, F and M and a new role name s. To obtain $\mathcal{T}'$, first replace in $\mathcal{T}$

- every $A_i \equiv \top$ with $A_i \equiv D \sqcup M$;
- every $A_i \equiv P$ with $A_i \equiv (P \sqcap D) \sqcup M$;
- every $A_i \equiv \neg A_j$ with $A_i \equiv (\overline{A}_j \sqcap D) \sqcup M$;
- every $A_i \equiv B_1 \sqcap B_2$ with $A_i \equiv (B_1 \sqcap B_2 \sqcap D) \sqcup M$;
- every $A_i \equiv \exists r.B$ with $A_i \equiv (D \sqcap \exists r.(B \sqcap D)) \sqcup M$

and then add

$$\begin{array}{lcl} M & \equiv & \exists r.M \sqcup \bigsqcup_{1 \leq i \leq n} (\overline{A_i} \sqcap A_i) \\ E_i & \sqsubseteq & (D \sqcap \exists s.(D \sqcap (E_1 \sqcup E_2))), \text{ for } i = 1, 2 \\ F & \equiv & (E_1 \sqcup E_2) \sqcap M. \end{array}$$

Intuitively, M plays the role of a marker that is set whenever there is an 'inconsistency' in the sense that a concept name A_i and its 'negation' $\overline{A}_i$ are both true at the same element. By the second last CI, we have $\mathcal{T} \models E_i \sqsubseteq \exists s^n.\top$ for $i \in \{1, 2\}$ which results in $\mathcal{T}'$ not being $\mathcal{EL}$-approximable when $\mathcal{T}$ is satisfiable. If $\mathcal{T}$ is not satisfiable, then the inconsistency marker M is true at the relevant elements in every model, which means that the above $\mathcal{EL}$-consequences can be approximated by $E_i \sqsubseteq \exists s.F$. The overall $\mathcal{EL}$-approximation of $\mathcal{T}'$ contains additional concept inclusions, as analyzed in more detail in the appendix of the long version. ❏

9 Future Work

Two interesting research directions are to exploit the techniques introduced in this paper for the actual computation of interpolants and approximants, and to extend the decision procedure for the existence of $\mathcal{EL}$-aproximants from classical $\mathcal{ELU}$-TBoxes to more expressive DLs and TBox formalisms.

Regarding the first point, we conjecture that EAs that represent uniform interpolants/approximants can be exploited to compute small interpolants/approximants, if they exist. A refinement of the proof of Proposition 23 given in the long version should be a promising starting point.

Regarding the second point, note that a naive adaptation of our decision procedure to TBoxes formulated in expressive DLs such as $\mathcal{ALC}$ does not work. For example, the $\mathcal{ALC}$-TBox $\mathcal{T} = \{A \sqsubseteq \forall r.B\}$ does not have an $\mathcal{EL}$-approximant. To see this, note that $\mathcal{T}$ has

$$A \sqcap \exists r.C \sqsubseteq \exists r.(B \sqcap C)$$

as an $\mathcal{EL}$-consequence, for any $\mathcal{EL}$-concept C. In particular, we thus have as a consequence the $\mathcal{EL}$-inclusion

$$A \sqcap \exists r.\exists s^n.E \sqsubseteq \exists r.(B \sqcap \exists s^m.E)$$

iff $n = m$, where E and s are fresh concept and role names. Semantically, this means that, for every rs^n-path in an interpretation $\mathcal{I}$ that starts at an element $d \in A^\mathcal{I}$, there must exist a (potentially different) rs^n-path that starts at d and satisfies B at the second element. It is standard to show that such languages, which involve 'unbounded counting', are not recognizable by an APTA (with parity/Rabin acceptance condition) and, therefore, not by an EA either.

Interestingly, a similar effect can be observed already for general $\mathcal{ELU}$-TBoxes. As an example, consider

$$\begin{array}{rcl} \mathcal{T}' = \{A \sqcap \exists r.\overline{B} & \sqsubseteq & \exists r.(B \sqcap (S \sqcup S')) \\ (S \sqcup S') & \sqsubseteq & \exists s.(S \sqcup S') \\ D & \sqsubseteq & B \sqcup \overline{B}\ \} \end{array}$$

$\mathcal{T}'$ does not have an $\mathcal{EL}$-approximant, for similar reasons as the above $\mathcal{ALC}$-TBox $\mathcal{T}$. As a hint, note that the $\mathcal{EL}$-inclusion

$$A \sqcap \exists r.(D \sqcap \exists s^n.E) \sqsubseteq \exists r.(B \sqcap \exists s^m.E)$$

is a consequence of $\mathcal{T}$ if, and only if, $n = m$. Despite these difficulties, one can well imagine EAs as a central ingredient of a more intricate procedure for checking $\mathcal{EL}$-approximability of general $\mathcal{ELU}$- or even $\mathcal{ALC}$-TBoxes, as a second step after first checking recognizability by APTAs.

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A Proofs for Section 3

We start by supplying proofs for some examples. Recall that $\mathcal{T}_1$ is given by

$$\begin{array}{rcl} \mathsf{Patient} \sqcap \exists \mathsf{finding}.\mathsf{MajorFinding} & \sqsubseteq & \mathsf{InPatient} \\ \mathsf{Finding} \sqcap \exists \mathsf{status}.\mathsf{PotentiallyLethal} & \sqsubseteq & \mathsf{MajorFinding} \\ \mathsf{MajorFinding} & \sqsubseteq & \mathsf{Finding} \end{array}$$

Proposition 34. *A uniform $\mathcal{EL}_\Sigma$-interpolant of $\mathcal{T}_1$ is given by the TBox $\mathcal{T}_1'$ consisting of:*

$$\mathsf{Patient} \sqcap \exists \mathsf{finding}.(\mathsf{Finding} \sqcap \exists \mathsf{status}.\mathsf{PotentiallyLethal}) \sqsubseteq \mathsf{InPatient}.$$

Proof. We apply Theorem 5. Thus, it is sufficient to show that $\mathsf{mod}(\mathcal{T}_1') = \mathsf{cl}^\Sigma_\approx(\mathsf{mod}(\mathcal{T}_1))$. The inclusion "$\supseteq$" is trivial. Conversely, assume that $\mathcal{I}$ is a model of $\mathcal{T}_1'$. Define $\mathcal{J}$ as $\mathcal{I}$ with the exception that

$$\mathsf{MajorFinding}^\mathcal{J} = (\mathsf{Finding} \sqcap \exists \mathsf{status}.\mathsf{PotentiallyLethal})^\mathcal{I}$$

It is readily checked that $\mathcal{J}$ is a model of $\mathcal{T}_1$ and, clearly, $(\mathcal{I}, d) \approx_\Sigma (\mathcal{J}, d)$ for all $d \in \Delta^\mathcal{I}$. Thus, $\mathcal{I} \in \mathsf{cl}^\Sigma_\approx(\mathsf{mod}(\mathcal{T}_1))$, as required. ❑

To compare uniform interpolants in $\mathcal{EL}$ with uniform interpolants in $\mathcal{ALC}$ we consider the $\mathcal{EL}$-TBox $\mathcal{T}_2$ defined as

$$A \sqsubseteq \exists r.X \qquad X \sqcap B \sqsubseteq \exists s.Y \qquad Y \sqsubseteq \exists s.Y$$

The proof of the following result can also be used to show the claim that $\{A \sqsubseteq \exists r.B, B \sqsubseteq \exists r.B\}$ does not have a uniform $\mathcal{EL}_{\{A,r\}}$-interpolant.

Proposition 35. *$\mathcal{T}_2$ does not have a uniform $\mathcal{EL}_\Sigma$-interpolant, for $\Sigma = \{A, B, X, r, s\}$.*

Proof. We apply Theorems 20 and 24.

Let $m > 0$ be arbitrary. Let $\mathcal{I}_1$ be defined by setting

- $\Delta^{\mathcal{I}_1} = \{0, 1, \ldots\}$,
- $A^{\mathcal{I}_1} = \emptyset$,
- $X^{\mathcal{I}_1} = B^{\mathcal{I}_1} = \{0\}$,
- $r^{\mathcal{I}_1} = \emptyset$, and
- $s^{\mathcal{I}_1} = \{(0,1), (1,2), \ldots\}$

and let $\mathcal{I}_2$ be defined by

- $\Delta^{\mathcal{I}_2} = \{0, \ldots, m, m+1\}$,
- $A^{\mathcal{I}_2} = \emptyset$,
- $X^{\mathcal{I}_2} = B^{\mathcal{I}_2} = \{0\}$,
- $r^{\mathcal{I}_2} = \emptyset$, and
- $s^{\mathcal{I}_2} = \{(0,1), \ldots, (m, m+1)\}$

Then

1. $\mathcal{I}_1^{\leq m} = \mathcal{I}_2^{\leq m}$,
2. $\mathcal{I}_1 \in L(\mathcal{A}_{\mathcal{T}_2,\Sigma})$,
3. $\mathcal{I}_2 \notin L(\mathcal{A}_{\mathcal{T}_2,\Sigma})$,
4. $\mathcal{I}_2(1) \in L(\mathcal{A}_{\mathcal{T}_2,\Sigma})$.

Thus, for every $m > 0$ Condition 2 of Theorem 20 is satisfied. It follows that no $\mathcal{EL}_\Sigma$-TBox is $\mathcal{EL}_\Sigma$-inseparable from $\mathcal{T}_1$. ❑

Proposition 36. *For $\Sigma' = \{A, B, r, s\}$, the TBox $\mathcal{T}_2' = \{A \sqsubseteq \exists r.\top\}$ is a uniform $\mathcal{EL}_{\Sigma'}$-interpolant of $\mathcal{T}_2$.*

Proof. We use Theorem 5. Thus, it is sufficient to show that $\mathsf{mod}(\mathcal{T}_2') = \mathsf{cl}^{\Sigma'}_\approx(\mathsf{mod}(\mathcal{T}_2))$. The inclusion "$\supseteq$" is trivial. Let $\mathcal{I}$ be a model of $\mathcal{T}_2$. We may assume that $\mathcal{I}$ only interprets Σ'. For any $d \in A^\mathcal{I}$ take a fresh n_d and define a new interpretation $\mathcal{J}$ by adding (d, n_d) to $r^\mathcal{I}$ and setting $X^\mathcal{I} = \{n_d \mid d \in A^\mathcal{I}\}$. Then $\mathcal{J}$ is a model of $\mathcal{T}_2$ and $(\mathcal{I}, d) \approx_\Sigma (\mathcal{J}, d)$ for all $d \in \Delta^\mathcal{I}$. ❑

In contrast, one can readily checked that there is no uniform $\mathcal{ALC}_{\Sigma'}$-interpolant of $\mathcal{T}_2$. Intuitively, because the infinite set of $\mathcal{ALC}_\Sigma$-CIs $\{A \sqcap \forall r.B \sqsubseteq \exists s^n.\top \mid n \geq 1\}$ is entailed by $\mathcal{T}_2$.

B Proofs for Section 4

Let $\mathcal{T}$ consist of

$$\begin{array}{rcl} \mathsf{Node} & \sqsubseteq & \mathsf{Left} \sqcup \mathsf{Right} \\ \mathsf{Left} & \sqsubseteq & \exists \mathsf{succ}.(\mathsf{Left} \sqcup \mathsf{Right}) \\ \mathsf{Right} & \sqsubseteq & \exists \mathsf{succ}.(\mathsf{Left} \sqcup \mathsf{Right}) \end{array}$$

and $\mathcal{T}_0$ be $\mathcal{T}$ plus

$$M = \{\mathsf{Left} \sqsubseteq \mathsf{Node}, \mathsf{Right} \sqsubseteq \mathsf{Node}\}.$$

Finally, define $\mathcal{T}_0'$ as

$$M \cup \{\mathsf{Node} \sqsubseteq \exists \mathsf{succ}.\mathsf{Node}\}.$$

Lemma 37. *$\mathcal{T}_0'$ is an $\mathcal{EL}$-approximant of $\mathcal{T}_0$.*

Proof. We show that $\mathcal{T}_0' \equiv^{\mathcal{EL}} \mathcal{T}_0$.
"$\mathcal{T}_0' \subseteq^{\mathcal{EL}} \mathcal{T}_0$". We show the contrapositive. Suppose the $\mathcal{EL}$-CI $C \sqsubseteq D$ does not follow from $\mathcal{T}_0$, i.e., $\mathcal{T}_0 \not\models C \sqsubseteq D$. We need to show that $\mathcal{T}_0' \not\models C \sqsubseteq D$. Now since $\mathcal{T}_0 \not\models C \sqsubseteq D$, there is some model $\mathcal{I}$ of $\mathcal{T}_0$ with $\mathcal{I} \not\models C \sqsubseteq D$. If we show that $\mathcal{I}$ is a model of $\mathcal{T}_0'$ then we immediately obtain that $\mathcal{T}_0' \not\models C \sqsubseteq D$. Hence our aim is to show that $\mathcal{I} \models \mathcal{T}_0'$. To this aim, we distinguish all the CIs that appear in $\mathcal{T}_0'$. By the definition of $\mathcal{I}$, we already have $\mathcal{I} \models \mathsf{Left} \sqsubseteq \mathsf{Node}$ and $\mathcal{I} \models \mathsf{Right} \sqsubseteq \mathsf{Node}$. It remains to show that $\mathcal{I} \models \mathsf{Node} \sqsubseteq \exists \mathsf{succ}.\mathsf{Node}$.

Let $d \in \mathsf{Node}^\mathcal{I}$. Then by $\mathsf{Node} \sqsubseteq \mathsf{Left} \sqcup \mathsf{Right} \in \mathcal{T}_0$, either $d \in \mathsf{Left}^\mathcal{I}$ or $d \in \mathsf{Right}^\mathcal{I}$. We only show it for the case that the former holds, the other case can be shown analogously. Then by $\mathsf{Left} \sqsubseteq \exists \mathsf{succ}.(\mathsf{Left} \sqcup \mathsf{Right})$, there is some $e \in \Delta^\mathcal{I}$ such that $(d, e) \in \mathsf{succ}^\mathcal{I}$ and $e \in (\mathsf{Left} \sqcup \mathsf{Right})^\mathcal{I}$. From the former, we obtain by $\mathsf{Left} \sqsubseteq \mathsf{Node}, \mathsf{Right} \sqsubseteq \mathsf{Node} \in \mathcal{T}_0$ that $e \in \mathsf{Node}^\mathcal{I}$. Then by $(d, e) \in \mathsf{succ}^\mathcal{I}$, $d \in (\exists \mathsf{succ}.\mathsf{Node})^\mathcal{I}$. Hence $\mathcal{I} \models \mathsf{Node} \sqsubseteq \exists \mathsf{succ}.\mathsf{Node}$.

"$\mathcal{T}_0 \subseteq^{\mathcal{EL}} \mathcal{T}_0'$". We show the contrapositive. Suppose the $\mathcal{EL}$-CI $C \sqsubseteq D$ does not follow from $\mathcal{T}_0'$, i.e., $\mathcal{T}_0' \not\models C \sqsubseteq D$. We need to show that $\mathcal{T}_0 \not\models C \sqsubseteq D$. We proceed towards contradiction so suppose that $\mathcal{T}_0 \models C \sqsubseteq D$.

Let $\mathcal{I}_0$ be a model of $\mathcal{T}_0'$, i.e., $\mathcal{I}_0 \models \mathcal{T}_0'$. W.l.o.g. we assume that $\mathcal{I}_0$ is a tree interpretation, $\rho_0 \in C^{\mathcal{I}_0} \setminus D^{\mathcal{I}_0}$, where ρ_0 is the root of $\mathcal{I}_0$, and $X^{\mathcal{I}_0} = \emptyset$ for all $X \notin \mathsf{sig}(\mathcal{T})$. Our aim is to construct a sequence of interpretations $\varsigma =$

$\mathcal{I}_0, \mathcal{I}_1, \ldots$, where $\mathcal{I}_{i+1}$ is obtained from $\mathcal{I}_i$ by fixing a 'minimal defect' in $\mathcal{I}_i$. To be more precise, we need the following definitions.

Let $\mathcal{I}$ be an interpretation in the sequence ς. A *defect* in $\mathcal{I}$ is a $d \in \mathsf{Node}^{\mathcal{I}}$ with $d \notin (\mathsf{Left} \sqcup \mathsf{Right})^{\mathcal{I}}$. A *repair* for the defect d consists of two copies $\mathcal{I}_1, \mathcal{I}_2$ of $\mathcal{I}$ which coincide with $\mathcal{I}$ with the exception that

- $\mathsf{Left}^{\mathcal{I}_1} = \mathsf{Left}^{\mathcal{I}} \cup \{d\}$;
- $\mathsf{Right}^{\mathcal{I}_2} = \mathsf{Right}^{\mathcal{I}} \cup \{d\}$.

A *minimal defect* is a defect d such that there is no defect on the path from the root of $\mathcal{I}$ to d.

Claim 1. Let d_0 be a minimal defect in $\mathcal{I}$ and $\mathcal{I}_1, \mathcal{I}_2$ its repairs. Then for all $d \in \Delta^{\mathcal{I}}$, $(\mathcal{I}, d) \approx (\mathcal{I}_1 \times \mathcal{I}_2, (d, d))$.

Let $\mathcal{I}, \mathcal{I}_1, \mathcal{I}_2$, and d_0 be as specified in the lemma.
"$\Rightarrow$" Define $S = \{(d, (d, d)) \mid d \in \Delta^{\mathcal{I}}\}$. We claim that for all $d \in \Delta^{\mathcal{I}}$, $S : (\mathcal{I}, d) \leq (\mathcal{I}_1 \times \mathcal{I}_2, (d, d))$. By $\Delta^{\mathcal{I}} = \Delta^{\mathcal{I}_1} = \Delta^{\mathcal{I}_2}$, we have $S \subseteq \Delta^{\mathcal{I}} \times (\Delta^{\mathcal{I}_1} \times \Delta^{\mathcal{I}_2})$. Moreover, for every $d \in \Delta^{\mathcal{I}}$, $(d, (d, d)) \in S$. It remains to show that (base) and (forth) are satisfied. Let $(d, (d_1, d_2)) \in S$.

- (base). Suppose $d \in A^{\mathcal{I}}$. By the definition of S, we have $d_1 = d_2 = d$ and thus, we need to show that $(d, d) \in A^{\mathcal{I}_1 \times \mathcal{I}_2}$. $d \in A^{\mathcal{I}}$ implies by the definition of $\mathcal{I}_1$ and $\mathcal{I}_2$ that $d \in A^{\mathcal{I}_1}$ and $d \in A^{\mathcal{I}_2}$. Then by the definition of $\mathcal{I}_1 \times \mathcal{I}_2$, $(d, d) \in A^{\mathcal{I}_1 \times \mathcal{I}_2}$.
- (forth). Suppose $(d, e) \in r^{\mathcal{I}}$. By the definition of S, we have $d_1 = d_2 = d$. We need to show that there is some $(e_1, e_2) \in \Delta^{\mathcal{I}_1 \times \mathcal{I}_2}$ such that $((d, d), (e_1, e_2)) \in r^{\mathcal{I}_1 \times \mathcal{I}_2}$ and $(e, (e_1, e_2)) \in S$. Since $\{(e, e)\} = \{(e_1, e_2) \mid (e, (e_1, e_2)) \in S\}$, it is enough to show that $((d, d), (e, e)) \in r^{\mathcal{I}_1 \times \mathcal{I}_2}$. $(d, e) \in r^{\mathcal{I}}$ implies by the definition of $\mathcal{I}_1$ and $\mathcal{I}_2$ that $(d, e) \in r^{\mathcal{I}_1} = r^{\mathcal{I}_2}$. Then by the definition of $\mathcal{I}_1 \times \mathcal{I}_2$, $((d, d), (e, e)) \in r^{\mathcal{I}_1 \times \mathcal{I}_2}$.

Hence it follows that for all $d \in \Delta^{\mathcal{I}}$, we have $S : (\mathcal{I}, d) \leq (\mathcal{I}_1 \times \mathcal{I}_2, (d, d))$.

"$\Leftarrow$" Let $S \subseteq (\Delta^{\mathcal{I}_1} \times \Delta^{\mathcal{I}_2}) \times \Delta^{\mathcal{I}}$ consist of all pairs $((d_1, d_2), d)$ such that

- $\mathsf{level}(d) = \mathsf{level}(d_1) = \mathsf{level}(d_2)$;
- if d_1 is on the path to d_0, then $d_2 = d$;
- if d_2 is on the path to d_0, then $d_1 = d$;
- if neither d_1 nor d_2 are on the path to d_0, then $d \in \{d_1, d_2\}$.

We claim that for all $d \in \Delta^{\mathcal{I}}$, $S : (\mathcal{I}_1 \times \mathcal{I}_2, (d, d)) \leq (\mathcal{I}, d)$. Let $d \in \Delta^{\mathcal{I}}$. By the definition of S, we have $((d, d), d) \in S$. It remains to show that (base) and (forth) are satisfied. Let $((d_1, d_2), d) \in S$.

- (base) Suppose $(d_1, d_2) \in A^{\mathcal{I}_1 \times \mathcal{I}_2}$. Then by the definition of a product, $d_1 \in A^{\mathcal{I}_1}$ and $d_2 \in A^{\mathcal{I}_2}$; and by the definition of S, $\mathsf{level}(d) = \mathsf{level}(d_1) = \mathsf{level}(d_2)$. We distinguish cases.
  - d_1 is on the path to d_0. This means $d_2 = d$. We proceed towards contradiction so suppose that $d \notin A^{\mathcal{I}}$. By the definition of $\mathcal{I}_2$ and $d \in A^{\mathcal{I}_2}$, this means that $d = d_0$ and $A = \mathsf{Right}$. Then by $d_1 \in A^{\mathcal{I}_1}$, we obtain $d_1 \in \mathsf{Right}^{\mathcal{I}_1}$. Since $\mathsf{Right}^{\mathcal{I}_1} = \mathsf{Right}^{\mathcal{I}}$, $d_1 \in \mathsf{Right}^{\mathcal{I}}$. This implies that $d_1 \neq d_0$, since d_0 is a defect in $\mathcal{I}$ and thus, $d_0 \notin (\mathsf{Left} \sqcup \mathsf{Right})^{\mathcal{I}}$. Since d_1 is on the path to d_0, $d_1 \neq d_0$ implies that $\mathsf{level}(d_1) < \mathsf{level}(d_0)$. But this contradicts with $\mathsf{level}(d) = \mathsf{level}(d_1)$. Hence $d \in A^{\mathcal{I}}$.
  - d_2 is on the path to d_0. Analogous to the previous case.
  - Neither d_1 nor d_2 are on the path to d_0. This implies that $d_1 \neq d_0$ and $d_2 \neq d_0$. By assumption we have that $d \in \{d_1, d_2\}$. Suppose first that $d = d_1$. Then by $d_1 \in A^{\mathcal{I}_1}$ and $d_1 \neq d_0$, we have $d_1 \in A^{\mathcal{I}}$. Finally by $d = d_1$, $d \in A^{\mathcal{I}}$. The case for $d = d_2$ can be shown analogously.

  Hence we conclude that (base) is satisfied.
- (forth) Suppose $((d_1, d_2), (e_1, e_2)) \in r^{\mathcal{I}_1 \times \mathcal{I}_2}$. We need to show that there is some $e \in \Delta^{\mathcal{I}}$ such that $(d, e) \in r^{\mathcal{I}}$ and $((e_1, e_2), e) \in S$. By $((d_1, d_2), (e_1, e_2)) \in r^{\mathcal{I}_1 \times \mathcal{I}_2}$, we obtain $(d_1, e_1) \in r^{\mathcal{I}_1}$ and $(d_2, e_2) \in r^{\mathcal{I}_2}$. Since $r^{\mathcal{I}} = r^{\mathcal{I}_1} = r^{\mathcal{I}_2}$, we have that $(d_1, e_1) \in r^{\mathcal{I}}$ and $(d_2, e_2) \in r^{\mathcal{I}}$. We distinguish cases.
  - d_1 is on the path to d_0. Then $d_2 = d$. By $(d_2, e_2) \in r^{\mathcal{I}}$, we have $(d, e_2) \in r^{\mathcal{I}}$. If we show that $((e_1, e_2), e_2) \in S$ then we are done. Now by $(d_1, e_1) \in r^{\mathcal{I}}$ and the fact that d_1 is on the path to d_0, we have the following cases.
    - e_1 is also on the path to d_0. Then by the definition of S, it immediately follows that $((e_1, e_2), e_2) \in S$, which is what we wanted to show.
    - e_1 is not on the path to d_0 because $d_1 = d_0$. Then by $\mathsf{level}(d_1) = \mathsf{level}(d_2) = \mathsf{level}(d_0)$ and $(d_2, e_2) \in r^{\mathcal{I}}$, we have that $\mathsf{level}(e_2) = \mathsf{level}(d_0) + 1$. Hence e_2 is also not on the path to d_0. Since neither e_1 nor e_2 are on the path to d_0, we obtain by the definition of S that $((e_1, e_2), e_2) \in S$, which is what we wanted to show.
  - d_2 is on the path to d_0. Analogous to the previous case.
  - Neither d_1 nor d_2 are on the path to d_0. Then $d \in \{d_1, d_2\}$. First suppose that $d = d_1$. This implies by $(d_1, e_1) \in r^{\mathcal{I}}$ that $(d, e_1) \in r^{\mathcal{I}}$. If we show that $((e_1, e_2), e_1) \in S$ then we are done. By the fact that for $i \in \{1, 2\}$, d_i is not on the path to d_0 and $(d_i, e_i) \in r^{\mathcal{I}}$, e_i is also not on the path to d_0. But then $((e_1, e_2), e_1) \in S$, which is what we wanted to show. The case for $d = d_2$ can be shown analogously.

  Hence we conclude that (forth) is satisfied.

This marks the end of the proof of Claim 1. Now for the proof of this direction of the lemma, we proceed as follows. By our assumption, we have that $\rho_0 \in C^{\mathcal{I}_0} \setminus D^{\mathcal{I}_0}$, where ρ_0 is the root of $\mathcal{I}_0$, i.e., $\mathcal{I}_0 \not\models C \sqsubseteq D$. We start from $\mathcal{I}_0$ and do the following. We choose a minimal defect d_0 in $\mathcal{I}_0$, if it has any. Let $\mathcal{I}_a$ and $\mathcal{I}_b$ be the repairs of this defect. By Claim 1, for all $d \in \Delta_0^{\mathcal{I}}$, we have that $(\mathcal{I}_0, d) \approx (\mathcal{I}_a \times \mathcal{I}_b, (d, d))$. By $\rho_0 \in C^{\mathcal{I}_0} \setminus D^{\mathcal{I}_0}$, this implies $(\rho_0, \rho_0) \in C^{\mathcal{I}_a \times \mathcal{I}_b} \setminus D^{\mathcal{I}_a \times \mathcal{I}_b}$. Hence by Lemma 8, we obtain $\rho_0 \in C^{\mathcal{I}_a}$, $\rho_0 \in C^{\mathcal{I}_b}$, and either $\rho_0 \notin D^{\mathcal{I}_a}$ or $\rho_0 \notin D^{\mathcal{I}_b}$. Let $c \in \{a, b\}$ such that $\rho_0 \notin D^{\mathcal{I}_c}$. Set $\mathcal{I}_1 = \mathcal{I}_c$. Obviously, $\mathcal{I}_1 \not\models C \sqsubseteq D$ and $\mathcal{I}_1$ lacks the defect d_0. Now we proceed inductively, just as the case from $\mathcal{I}_0$ to $\mathcal{I}_1$, to obtain the sequence $\varsigma = \mathcal{I}_0, \mathcal{I}_1, \ldots$. The interpretation $\mathcal{I}$ in the limit of this construction satisfies the following properties:

- $\Delta^{\mathcal{I}} = \Delta^{\mathcal{I}_0}$,
- $\mathsf{Left}^{\mathcal{I}} \supseteq \mathsf{Left}^{\mathcal{I}_0}$,
- $\mathsf{Right}^{\mathcal{I}} \supseteq \mathsf{Right}^{\mathcal{I}_0}$,
- $\mathsf{Left}^{\mathcal{I}} \cup \mathsf{Right}^{\mathcal{I}} = \mathsf{Node}^{\mathcal{I}_0}$,
- $\mathsf{Node}^{\mathcal{I}} = \mathsf{Node}^{\mathcal{I}_0}$,
- $\mathsf{succ}^{\mathcal{I}} = \mathsf{succ}^{\mathcal{I}_0}$,
- for all $P \in (\mathsf{N_C} \cup \mathsf{N_R}) \setminus \{\mathsf{Left}, \mathsf{Right}, \mathsf{Node}, \mathsf{succ}\}$, $P^{\mathcal{I}_0} = P^{\mathcal{I}}$.

As a consequence, we have the following.

Claim 2. $\mathcal{I} \models \mathcal{T}_0$.

We show that $\mathcal{I}$ satisfies every CI in $\mathcal{T}_0$.

- $\mathsf{Node} \sqsubseteq \mathsf{Left} \sqcup \mathsf{Right}$: Suppose $d \in \mathsf{Node}^{\mathcal{I}}$. By the definition of $\mathcal{I}$, we first obtain $d \in \mathsf{Node}^{\mathcal{I}_0}$, and again by the definition of $\mathcal{I}$, this implies $d \in \mathsf{Left}^{\mathcal{I}} \cup \mathsf{Right}^{\mathcal{I}}$, which is what we wanted to show.
- $\mathsf{Left} \sqsubseteq \exists \mathsf{succ}.(\mathsf{Left} \sqcup \mathsf{Right})$: Suppose $d \in \mathsf{Left}^{\mathcal{I}}$. Then by the definition of $\mathcal{I}$, $d \in \mathsf{Node}^{\mathcal{I}_0}$. Since $\mathcal{I}_0 \models \mathsf{Node} \sqsubseteq \exists \mathsf{succ}.\mathsf{Node}$, there is some $e \in \Delta^{\mathcal{I}_0}$ such that $(d,e) \in \mathsf{succ}^{\mathcal{I}_0}$ and $e \in \mathsf{Node}^{\mathcal{I}_0}$. Then by the definition of $\mathcal{I}$, $(d,e) \in \mathsf{succ}^{\mathcal{I}}$ and $e \in \mathsf{Left}^{\mathcal{I}} \cup \mathsf{Right}^{\mathcal{I}}$. Hence $d \in (\exists \mathsf{succ}.(\mathsf{Left} \sqcup \mathsf{Right}))^{\mathcal{I}}$.
- $\mathsf{Right} \sqsubseteq \exists \mathsf{succ}.(\mathsf{Left} \sqcup \mathsf{Right})$: Can be shown analogously to the previous case.
- $\mathsf{Left} \sqsubseteq \mathsf{Node}$: Suppose $d \in \mathsf{Left}^{\mathcal{I}}$. Then it immediately follows by the definition of $\mathcal{I}$ that $d \in \mathsf{Node}^{\mathcal{I}_0} = \mathsf{Node}^{\mathcal{I}}$.
- $\mathsf{Right} \sqsubseteq \mathsf{Node}$: Can be shown analogously to the previous item.

Hence we conclude that the claim holds.

Now by our inductive construction, we obtain that $\mathcal{I} \not\models C \sqsubseteq D$ and by Claim 2, we have $\mathcal{I} \models \mathcal{T}_0$. The latter and $\mathcal{T}_0 \models C \sqsubseteq D$ imply that $\mathcal{I} \models C \sqsubseteq D$. Hence we obtained the contradiction. Thus we conclude that $\mathcal{T}_0 \subseteq^{\mathcal{EL}} \mathcal{T}_0'$. ❑

Theorem 7 Let $\mathcal{T}$ be a (general) $\mathcal{ELU}$-TBox and $\mathcal{T}'$ the $\mathcal{EL}$-approximant of $\mathcal{T}$. If $\mathcal{T}''$ is a (general) $\mathcal{ELU}$-TBox with $\mathcal{T} \models \mathcal{T}''$ and answering instance queries w.r.t. $\mathcal{T}''$ is in PTIME, then $\mathcal{T}' \models \mathcal{T}''$ (unless PTIME=CONP).

Proof. Let $\mathcal{T}, \mathcal{T}''$ be (general) $\mathcal{ELU}$-TBoxes with $\mathcal{T} \models \mathcal{T}''$ and $\mathcal{T}'$ the $\mathcal{EL}$-approximant of $\mathcal{T}$. If $\mathcal{T}' \not\models \mathcal{T}''$, then $\mathcal{T}''$ is not equivalent to any $\mathcal{EL}$-TBox. Thus, there exist $\mathcal{EL}$-concepts $C, D_1, \ldots, D_n$ such that $\mathcal{T}'' \models C \sqsubseteq D_1 \sqcup \cdots D_n$ but $\mathcal{T}'' \not\models C \sqsubseteq D_i$ for $1 \leq i \leq n$. Then $\mathcal{T}''$ does not have the ABox disjunction property: there exists an ABox $\mathcal{A}$ and an individual a such that $\mathcal{T}, \mathcal{A} \models (D_1 \sqcup \cdots \sqcup D_n)(a)$ but $\mathcal{T}, \mathcal{A} \not\models D_i(a)$ for all $1 \leq i \leq n$. (For $\mathcal{A}$ one can take the ABox corresponding to the $\mathcal{EL}$-concept C.) But then, as shown in (Lutz and Wolter 2012), instance checking w.r.t. $\mathcal{T}$ is coNP-hard. ❑

Towards Lemma 10 We now prove a sequence of lemmas which are required to show the implication from Point 1 to Point 2 in Lemma 10.

An interpretation $\mathcal{I}$ is *$\mathcal{EL}$-saturated* if it satisfies the following condition, for all $r \in \mathsf{N_R}$: if $d \in \Delta^{\mathcal{I}}$ and Γ is a (potentially infinite) set of $\mathcal{EL}$-concepts such that, for all finite $\Psi \subseteq \Gamma$, $d \in (\exists r. \bigsqcap \Psi)^{\mathcal{I}}$, then there is an e with $(d,e) \in r^{\mathcal{I}}$ and $e \in \Gamma^{\mathcal{I}}$ (here we write $d \in \Gamma^{\mathcal{I}}$ if for all $C \in \Gamma$, we have that $d \in C^{\mathcal{I}}$). Observe that all interpretations with finite outdegree are $\mathcal{EL}$-saturated. We set $(\mathcal{I}_1, d_1) \subseteq_\Sigma (\mathcal{I}_2, d_2)$ if $d_1 \in C^{\mathcal{I}_1}$ implies $d_2 \in C_2^{\mathcal{I}}$ for all $\mathcal{EL}_\Sigma$-concepts C.

Lemma 38. *For all $\mathcal{EL}$-saturated pointed interpretations $(\mathcal{I}_1, d_1)$, $(\mathcal{I}_2, d_2)$ and all signatures Σ, we have that*

$$(\mathcal{I}_1, d_1) \equiv_\Sigma (\mathcal{I}_2, d_2) \textit{ implies } (\mathcal{I}_1, d_2) \approx_\Sigma (\mathcal{I}_2, d_2).$$

Proof. Suppose that $(\mathcal{I}_1, d_1) \equiv_\Sigma (\mathcal{I}_2, d_2)$ and that $\mathcal{I}_1, \mathcal{I}_2$ are $\mathcal{EL}$-saturated. Define the following relations:

$$\begin{aligned} S &= \{(e_1, e_2) \in \Delta^{\mathcal{I}_1} \times \Delta^{\mathcal{I}_2} \mid (\mathcal{I}_1, e_1) \subseteq_\Sigma (\mathcal{I}_2, e_2)\} \\ S' &= \{(e_2, e_1) \in \Delta^{\mathcal{I}_2} \times \Delta^{\mathcal{I}_1} \mid (\mathcal{I}_2, e_2) \subseteq_\Sigma (\mathcal{I}_1, e_1)\} \end{aligned}$$

Claim. S is a Σ-simulation between $(\mathcal{I}_1, d_1)$ and $(\mathcal{I}_2, d_2)$.

Proof of claim. Since $(\mathcal{I}_1, d_1) \equiv_\Sigma (\mathcal{I}_2, d_2)$, we have by the construction of S that $(d_1, d_2) \in S$. Thus, it remains to show that S also satisfies (base) and (forth).

- (base). Suppose $A \in \Sigma \cap N_C$, $e_1 \in A^{\mathcal{I}}$, and $(e_1, e_2) \in S$. Then by the construction of S, it immediately follows that $e_2 \in A^{\mathcal{J}}$.
- (forth). Suppose $r \in \Sigma \cap N_R$, $(e_1, e_1') \in r^{\mathcal{I}_1}$, and $(e_1, e_2) \in S$. Our aim is to show that there is some $e_2' \in \Delta^{\mathcal{I}_2}$ such that $(e_1', e_2') \in r^{\mathcal{I}_2}$ and $(e_2, e_2') \in S$. Let Γ be the set of all Σ-concepts C such that $e_1' \in C^{\mathcal{I}_1}$. Since $\mathcal{I}_1$ is $\mathcal{EL}$-saturated, for every finite $\Psi \subseteq \Gamma$, $e_1 \in (\exists r. \bigsqcap \Psi)^{\mathcal{I}_1}$. By the construction of S, this implies $e_2 \in (\exists r. \bigsqcap \Psi)^{\mathcal{I}_2}$. Since $\mathcal{I}_2$ is $\mathcal{EL}$-saturated, there is some $e_2' \in \Delta^{\mathcal{I}_2}$ such that $(e_2, e_2') \in r^{\mathcal{I}_2}$ and $e_2' \in \Gamma^{\mathcal{I}_2}$. This means $(\mathcal{I}_1, e_1') \subseteq_\Sigma (\mathcal{I}_2, e_2')$ and thus $(e_1', e_2') \in S$ which is what we wanted to show.

One can show in the same way that S' is a Σ-simulation between $(\mathcal{I}_2, d_2)$ and $(\mathcal{I}_1, d_1)$. In conclusion, $(\mathcal{I}_1, d_1)$ and $(\mathcal{I}_2, d_2)$ are Σ-equisimilar. ❑

We require one more lemma.

Lemma 39. *If $\mathcal{I}_i, i \in I$, are $\mathcal{EL}$-saturated, then $\mathcal{I} = \prod_{i \in I} \mathcal{I}_i$ is $\mathcal{EL}$-saturated.*

Proof. Assume $\mathcal{I}_i, i \in I$, are $\mathcal{EL}$-saturated and Γ is a set of $\mathcal{EL}$-concepts. Let $\vec{d} = (d_i)_{i \in I} \in \Delta^{\mathcal{I}}$ such that for all finite $\Psi \subseteq \Gamma$, $\vec{d} \in (\exists r. \bigsqcap \Psi)^{\mathcal{I}}$. Then $d_i \in (\exists r. \bigsqcap \Psi)^{\mathcal{I}_i}$, for all $i \in I$ and finite $\Psi \subseteq \Gamma$. By $\mathcal{EL}$-saturatedness of all $\mathcal{I}_i, i \in I$, we obtain $e_i \in \Delta^{\mathcal{I}_i}$ with $(d_i, e_i) \in r^{\mathcal{I}_i}$ such that $e_i \in D^{\mathcal{I}_i}$ for all $i \in I$ and $D \in \Gamma$. Let $\vec{e} = (e_i)_{i \in I}$. Then $(\vec{d}, \vec{e}) \in r^{\mathcal{I}}$ and $\vec{e} \in D^{\mathcal{I}}$ for all $D \in \Gamma$, as required. ❑

Point 1 implies Point 2 in Lemma 10. Assume Point 1 holds for $\mathcal{I}$ and $d \in \Delta^{\mathcal{I}}$. Let

$$X = \{C \mid d \in C^{\mathcal{I}},\ C \text{ an } \mathcal{EL}\text{-concept}\}.$$

Denote by $\overline{X}$ the set of $\mathcal{EL}$-concepts not in X. By compactness of FO and Point 1, for every $C \in \overline{X}$, there exists a model $\mathcal{I}_C$ of $\mathcal{T}$ and a $d_C \in \Delta^{\mathcal{I}_C}$ with

- $d_C \in D^{\mathcal{I}_C}$ for all $D \in X$,
- $d_C \notin C^{\mathcal{I}_C}$.

We may assume that the $\mathcal{I}_C$, $C \in \overline{X}$ are $\mathcal{EL}$-saturated. Let $\mathcal{J} = \prod_{C \in \overline{X}} \mathcal{I}_C$. We have $(\mathcal{I}, d) \equiv (\mathcal{J}, \vec{d})$ for $\vec{d} = (d_C)_{C \in \overline{X}}$. Moreover, by Lemma 39, $\mathcal{J}$ is $\mathcal{EL}$-saturated. Thus, by Lemma 38, $(\mathcal{I}, d) \approx (\mathcal{J}, \vec{d})$ as required.

Theorem 9 is a consequence of Lemma 10 First assume that $\mathcal{T}'$ is an $\mathcal{EL}$-approximant of $\mathcal{T}$. Then $\mathcal{T} \equiv^{\mathcal{EL}} \mathcal{T}'$ and thus every $\mathcal{I} \in \mathsf{mod}(\mathcal{T}')_{\restriction \mathsf{finout}}$ satisfies all $\mathcal{EL}$-CIs $C \sqsubseteq D$ with $\mathcal{T} \models C \sqsubseteq D$, which yields $\mathcal{I} \in \mathsf{cl}_{\approx}(\mathsf{cl}_{\Pi}(\mathsf{mod}(\mathcal{T})))_{\restriction \mathsf{finout}}$ by Lemma 10; conversely, if $\mathcal{I} \notin \mathsf{mod}(\mathcal{T}')_{\restriction \mathsf{finout}}$, then $\mathcal{I} \not\models C \sqsubseteq D$ for some $\mathcal{EL}$-CI $C \sqsubseteq D \in \mathcal{T}'$. Since $\mathcal{T} \equiv^{\mathcal{EL}} \mathcal{T}'$ we have $\mathcal{T} \models C \sqsubseteq D$, which yields $\mathcal{I} \notin \mathsf{mod}(\mathcal{T})_{\restriction \mathsf{finout}} \subseteq \mathsf{cl}_{\approx}(\mathsf{cl}_{\Pi}(\mathsf{mod}(\mathcal{T})))_{\restriction \mathsf{finout}}$.

Now assume $\mathsf{mod}(\mathcal{T}')_{\restriction \mathsf{finout}} = \mathsf{cl}_{\approx}(\mathsf{cl}_{\Pi}(\mathsf{mod}(\mathcal{T})))_{\restriction \mathsf{finout}}$. We want to show that $\mathcal{T} \equiv^{\mathcal{EL}} \mathcal{T}'$ which boils down to showing that $\mathcal{T}'$ $\mathcal{EL}_{\Sigma}$-entails $\mathcal{T}$ since $\mathcal{T} \models \mathcal{T}'$. Assume that $\mathcal{T}' \not\models C \sqsubseteq D$. Then there is a model $\mathcal{I}$ of $\mathcal{T}$ with $\mathcal{I} \not\models C \sqsubseteq D$ and we can w.l.o.g. assume $\mathcal{I}$ to be of finite outdegree. We thus have $\mathcal{I} \in \mathsf{cl}_{\approx}(\mathsf{cl}_{\Pi}(\mathsf{mod}(\mathcal{T})))_{\restriction \mathsf{finout}}$ and thus there is a model $\mathcal{J} \in \mathsf{mod}(\mathcal{T})$ such that $\mathcal{I} \not\models C \sqsubseteq D$, which yields $\mathcal{T} \not\models C \sqsubseteq D$ as required.

C Proofs for Section 5

Before we prove the lemmas and theorems from Section 5, we establish some general technical machinery for working with $\mathcal{EL}$ automata.

Let $\mathcal{A} = (Q, P, \Sigma_N, \Sigma_E, \delta)$ be an EA and $\mathcal{I}$ an interpretation. The *canonical pre-run* $\rho_{\mathsf{c}}^{\mathcal{I}}$ *of* $\mathcal{A}$ *on* $\mathcal{I}$ is defined as the limit of a sequence of maps $\rho_{\mathsf{c},0}^{\mathcal{I}} \subseteq \rho_{\mathsf{c},1}^{\mathcal{I}} \subseteq \rho_{\mathsf{c},2}^{\mathcal{I}} \cdots$ from $\Delta^{\mathcal{I}}$ to 2^Q. Define $\rho_{\mathsf{c},0}^{\mathcal{I}}$ by setting for each $d \in \Delta^{\mathcal{I}}$,

$$\begin{aligned}\rho_{\mathsf{c},0}^{\mathcal{I}}(d) \quad = \quad & \{q \in Q \mid \mathsf{true} \rightarrow q \in \delta\} \cup \\ & \{q \in Q \mid A \rightarrow q \in \delta \wedge d \in A^{\mathcal{I}}\}.\end{aligned}$$

Now, $\rho_{\mathsf{c},i+1}^{\mathcal{I}}$ is defined as follows. Start with $\rho_{\mathsf{c},i}^{\mathcal{I}}$ and exhaustively apply Condition 3 of runs, viewed as a rule; call the result $\sigma_{\mathsf{c},i}^{\mathcal{I}}$; then $\rho_{\mathsf{c},i+1}^{\mathcal{I}}$ is the extension of $\sigma_{\mathsf{c},i}^{\mathcal{I}}$ obtained by adding $q \in Q$ to $\sigma_{\mathsf{c},i}^{\mathcal{I}}(d)$ whenever there is a $(d, e) \in r^{\mathcal{I}}$ and $\langle r \rangle q_1 \rightarrow q \in \delta$ with $q_1 \in \sigma_{\mathsf{c},i}^{\mathcal{I}}(e)$ (note: no repetition!). This finishes the definition of $\rho_{\mathsf{c}}^{\mathcal{I}}$.

For $d \in \Delta^{\mathcal{I}}$ and $p \in P$, a map $\sigma : \Delta^{\mathcal{I}} \rightarrow 2^P$ is a *witness for* p *at* d if

- $p \in \sigma(d)$
- Conditions 6 to 9 of runs are satisfied.

The canonical pre-run $\rho_{\mathsf{c}}^{\mathcal{I}}$ is *consistent* if the following conditions are satisfied:

- for all $d \in \Delta^{\mathcal{I}}$, $q \in \rho_{\mathsf{c}}^{\mathcal{I}}(d)$, and $q \rightarrow p \in \delta$, there is witness for p at d.

Lemma 40. *$\mathcal{I} \in L(\mathcal{A})$ iff the canonical pre-run of $\mathcal{A}$ on $\mathcal{I}$ is consistent.*

Proof. "$\Rightarrow$" Suppose $\mathcal{I} \in L(\mathcal{A})$. Then there is some run ρ of $\mathcal{A}$ on $\mathcal{I}$. Define the function $\sigma : \Delta^{\mathcal{I}} \rightarrow 2^P$ as follows:

$$\sigma(d) = \rho(d) \cap P.$$

We have that

$(*)$ σ satisfies Conditions 6 to 9 of runs

since ρ satisfies these conditions. Let ρ_c be the canonical pre-run of $\mathcal{A}$ on $(\mathcal{I}, d_0)$. It is easy to see that the following property holds:

- for all $d \in \Delta^{\mathcal{I}}$ and $q \in Q$, if $q \in \rho_c(d)$ then $q \in \rho(d)$.

Now let $d \in \Delta^{\mathcal{I}}$ and $q \in Q$ such that $q \in \rho_c(d)$ and $q \rightarrow p \in \delta$. By the property we established previously, $q \in \rho(d)$. Since ρ is a run, we have that $p \in \rho(d)$. Then by the definition of σ, we obtain $p \in \sigma(d)$. Moreover, by $(*)$, σ also satisfies Conditions 6 to 9 of runs. This means that σ is a witness for p at d. This implies that ρ_c is consistent since our choice of $d \in \Delta^{\mathcal{I}}$ and $q \in Q$ were arbitrary.

"$\Leftarrow$" Suppose that the canonical pre-run ρ_c of $\mathcal{A}$ on $\mathcal{I}$ is consistent. For a $d \in \Delta^{\mathcal{I}}$ and $p \in P$, we say that p *is a burden of* d if the following holds:

- there is some $q \in Q$ with $q \in \rho_c(d)$ and $q \rightarrow p \in \delta$.

We denote by B_d the set of all burdens of d. For a $p \in B_d$, σ_d^p denotes the witness of p at d. Such a witness always exists by our assumption about ρ_c. Moreover, by the definition of a witness, we have that $p \in \sigma_d^p(d)$. Now define the function $\rho : \Delta^{\mathcal{I}} \rightarrow 2^{Q \cup P}$ as follows:

$$\rho(d) = \rho_c(d) \cup \bigcup_{p \in B_d} \sigma_d^p(d)$$

We claim that ρ is a run of $\mathcal{A}$ on $(\mathcal{I}, d_0)$. This is a consequence of the fact that for every burden p of every $d \in \Delta^{\mathcal{I}}$, we have a witness σ_d^p and that σ_d^p satisfies Conditions 6-9 of runs. ❑

We now return to the proofs of Section 5. We split Theorem 15 into two separate theorems.

Theorem 41. *EA emptiness is in* PTIME.

Proof. We reduce EA emptiness to general TBox unsatisfiability in $\mathcal{EL}_{\bot}$, which is in PTIME (Baader, Brandt, and Lutz 2005). Let $\mathcal{A} = (Q, P, \Sigma_N, \Sigma_E, \delta)$ be an EA. For every $s \in Q \cup P$, let A_s be a distinct concept name that is not in Σ_N. Now define the function $\rightsquigarrow$ that translates transitions in δ to concept inclusions in $\mathcal{EL}$ as follows:

$$\begin{array}{rcl} \mathsf{true} \to q & \rightsquigarrow & \top \sqsubseteq A_q \\ A \to q & \rightsquigarrow & A \sqsubseteq A_q \\ q_1 \wedge \ldots \wedge q_n \to q & \rightsquigarrow & A_{q_1} \sqcap \ldots \sqcap A_{q_n} \sqsubseteq A_q \\ \langle r \rangle q_1 \to q & \rightsquigarrow & \exists r.A_{q_1} \sqsubseteq A_q \\ q \to p & \rightsquigarrow & A_q \sqsubseteq A_p \\ p \to p_1 & \rightsquigarrow & A_p \sqsubseteq A_{p_1} \\ p \to \langle r \rangle p_1 & \rightsquigarrow & A_p \sqsubseteq \exists r.A_{p_1} \\ p \to A & \rightsquigarrow & A_p \sqsubseteq A \\ p \to \mathsf{false} & \rightsquigarrow & A_p \sqsubseteq \bot. \end{array}$$

Let $\mathcal{T}_\mathcal{A} = \{C \sqsubseteq D \mid \varphi \in \delta \text{ and } \varphi \rightsquigarrow C \sqsubseteq D\}$. It remains to show the following.

Claim. $L(\mathcal{A}) = \emptyset$ iff $\mathcal{T}_\mathcal{A}$ is satisfiable.

"$\Rightarrow$". We show the contrapositive. Suppose that $\mathcal{T}_\mathcal{A}$ is satisfiable. Then there is a model $\mathcal{I}$ of $\mathcal{T}_\mathcal{A}$. We claim that $\mathcal{I} \in L(\mathcal{A})$, thus $L(\mathcal{A})$ is nonempty. Indeed, it is not hard to verify that the function $\rho : \Delta^\mathcal{I} \to 2^{Q \cup P}$ defined by setting

$$\rho(d) = \{s \in Q \cup P \mid d \in A_s^\mathcal{I}\} \quad \text{for all } d \in \Delta^\mathcal{I}$$

is a run of $\mathcal{A}$ on $\mathcal{I}$.

"$\Leftarrow$". Again we show the contrapositive. Suppose $L(\mathcal{A}) \neq \emptyset$ and take an interpretation $\mathcal{I} \in L(\mathcal{A})$. Let ρ be a run of $\mathcal{A}$ on $\mathcal{I}$. Extend $\mathcal{I}$ by setting

$${A_s}^\mathcal{I} = \{d \in \Delta^\mathcal{I} \mid s \in \rho(d)\} \text{ for all } s \in Q \cup P.$$

Using the construction of $\mathcal{T}_\mathcal{A}$ and the definition of runs, it is not hard to verify that $\mathcal{I} \models \mathcal{T}_\mathcal{A}$. Hence, $\mathcal{T}_\mathcal{A}$ is satisfiable. ❑

Theorem 42. *EA containment is* EXPTIME*-complete.*

The lower bound is a consequence of the fact that deciding $\mathcal{EL}$-conservative extensions is EXPTIME-hard (Lutz and Wolter 2010) and, by Theorem 28, can be reduced in polynomial time to containment of EAs. For the upper bound, we show that every EA can be translated in polytime into an equivalent Wilke automaton as defined in (Wilke 2001). This suffices since containment of Wilke automata is known to be in EXPTIME.

Definition 43 (APTA)**.** An *alternating parity tree automaton (APTA)* is a tuple $\mathcal{A} = (Q, \Sigma_N, \Sigma_E, q_0, \delta, \Omega)$, where Q is a finite set of *states*, $\Sigma_N \subseteq \mathsf{N_C}$ is the finite *node alphabet*, $\Sigma_E \subseteq \mathsf{N_R}$ is the finite *edge alphabet*, $q_0 \in Q$ is the *initial state*, $\delta : Q \to \mathsf{mov}(\mathcal{A})$, is the transition function with $\mathsf{mov}(\mathcal{A}) = \{\mathsf{true}, \mathsf{false}, A, \neg A, q, q \wedge q', q \vee q', \langle r \rangle q, [r]q \mid A \in \Sigma_N, q, q' \in Q, r \in \Sigma_E\}$ the set of *moves* of the automaton, and $\Omega : Q \to \mathbb{N}$ is the *priority function*.

Intuitively, the move q means that the automaton sends a copy of itself in state q to the element of the interpretation that it is currently processing, $\langle r \rangle q$ means that a copy in state q is sent to an r-successor of the current element, and $[r]q$ means that a copy in state q is sent to every r-successor.

It will be convenient to use arbitrary modal formulas in negation normal form when specifying the transition function of APTAs. The more restricted form required by Definition 43 can then be attained by introducing intermediate states. In subsequent constructions that involve APTAs, we will not describe those additional states explicitly. However, we will (silently) take them into account when stating size bounds for automata.

In what follows, a Σ*-labelled tree* is a pair (T, ℓ) with T a tree and $\ell : T \to \Sigma$ a node labelling function. A *path* π in a tree T is a subset of T such that $\varepsilon \in \pi$ and for each $x \in \pi$ that is not a leaf in T, π contains one son of x.

Definition 44 (Run)**.** Let $(\mathcal{I}, d_0)$ be a pointed $\Sigma_N \cup \Sigma_E$-interpretation and $\mathcal{A} = (Q, \Sigma_N, \Sigma_E, q_0, \delta, \Omega)$ an APTA. A *run* of $\mathcal{A}$ on $(\mathcal{I}, d_0)$ is a $Q \times \Delta^\mathcal{I}$-labelled tree (T, ℓ) such that $\ell(\varepsilon) = (q_0, d_0)$ and for every $x \in T$ with $\ell(x) = (q, d)$:

- $\delta(q) \neq \mathsf{false}$;
- if $\delta(q) = A$ ($\delta(q) = \neg A$), then $d \in A^\mathcal{I}$ ($d \notin A^\mathcal{I}$);
- if $\delta(q) = q' \wedge q''$, then there are sons y, y' of x with $\ell(y) = (q', d)$ and $\ell(y') = (q'', d)$;
- if $\delta(q) = q' \vee q''$, then there is a son y of x with $\ell(y) = (q', d)$ or $\ell(y') = (q'', d)$;
- if $\delta(q) = \langle r \rangle q'$, then there is a $(d, d') \in r^\mathcal{I}$ and a son y of x with $\ell(y) = (q', d')$;
- if $\delta(q) = [r]q'$ and $(d, d') \in r^\mathcal{I}$, then there is a son y of x with $\ell(y) = (q', d')$.

A run (T, ℓ) is *accepting* if for every path π of T, the maximal $i \in \mathbb{N}$ with $\{x \in \pi \mid \ell(x) = (q, d) \text{ with } \Omega(q) = i\}$ infinite is even. We use $L(\mathcal{A})$ to denote the language accepted by $\mathcal{A}$, i.e., the set of pointed $\Sigma_N \cup \Sigma_E$-interpretations $(\mathcal{I}, d)$ such that there is an accepting run of $\mathcal{A}$ on $(\mathcal{I}, d)$.

When working on a pointed interpretation $(\mathcal{I}, d_0)$, an APTA $\mathcal{A}$ can clearly not "see" points that are unreachable from d_0 along role edges in the edge alphabet Σ_E of $\mathcal{A}$. We shall thus concentrate on pointed interpretations $(\mathcal{I}, d_0)$ that are *rooted* and where any $d \in \Delta^\mathcal{I}$ is reachable from d_0 along a path of Σ_E-edges. For any EA or APTA $\mathcal{A}$, let $L^r(\mathcal{A})$ be $L(\mathcal{A})$ restricted to pointed interpretations $(\mathcal{I}, d_0)$ that are rooted. *EA containment over rooted models* means to decide, given EAs $\mathcal{A}_1$ and $\mathcal{A}_2$, whether $L^r(\mathcal{A}_1) \subseteq L^r(\mathcal{A}_2)$; likewise for *APTA containment over rooted models*.

Lemma 45. *EA containment coincides with EA containment over rooted models; and APTA containment coincides with APTA containment over rooted models.*

Proof. We split the proof of the lemma into two claims.

Claim 1. EA containment coincides with EA containment over rooted models.

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two EAs. First suppose that $L(\mathcal{A}_1) \not\subseteq L(\mathcal{A}_2)$. Then there is some interpretation $\mathcal{I}$ such that $\mathcal{I} \in L(\mathcal{A}_1)$ and $\mathcal{I} \notin L(\mathcal{A}_2)$. By Lemma 40, the canonical prerun ρ_c of $\mathcal{A}_2$ on $\mathcal{I}$ is not consistent. In other words, there is some $d_0 \in \Delta^\mathcal{I}$ and $q \to p \in \delta_2$ such that $q \in \rho_c(d_0)$ and there is no witness for p at d_0. Let $(\mathcal{J}, d_0)$ be the pointed interpretation obtained by unravelling $(\mathcal{I}, d_0)$ using only Σ_E-edges. Obviously, $(\mathcal{J}, d_0)$ is rooted. We observe that

1. $(\mathcal{J}, d_0) \in L^r(\mathcal{A}_1)$ since we can define a run ρ' of $\mathcal{A}_1$ on $(\mathcal{J}, d_0)$ using the run ρ of $\mathcal{A}_1$ on $\mathcal{I}$; and

2. $(\mathcal{J}, d_0) \notin L^r(\mathcal{A}_2)$ since the canonical pre-run ρ'_c of $\mathcal{A}_2$ on $(\mathcal{J}, d_0)$ can be shown to be inconsistent using the fact that ρ_c was inconsistent.

Hence $L^r(\mathcal{A}_1) \not\subseteq L^r(\mathcal{A}_2)$.

For the other direction, suppose that $L^r(\mathcal{A}_1) \not\subseteq L^r(\mathcal{A}_2)$. Then there is some pointed interpretation $(\mathcal{I}, d_0)$ such that $(\mathcal{I}, d_0) \in L^r(\mathcal{A}_1)$ and $(\mathcal{I}, d_0) \notin L^r(\mathcal{A}_2)$. Let ρ_1 be the run of $\mathcal{A}_1$ on $(\mathcal{I}, d_0)$. Since every $d \in \Delta^{\mathcal{I}}$ is reachable from d_0 along a path of Σ_E-edges, it follows that ρ_1 is a function with domain $\Delta^{\mathcal{I}}$. Then we immediately have that ρ_1 is a run of $\mathcal{A}_1$ on $\mathcal{I}$, i.e., $\mathcal{I} \in L(\mathcal{A}_1)$. Now let ρ_c be the canonical pre-run of $\mathcal{A}_2$ on $(\mathcal{I}, d_0)$. Since every $d \in \Delta^{\mathcal{I}}$ is reachable from d_0 along a path of Σ_E-edges, it follows that ρ_c is a function with domain $\Delta^{\mathcal{I}}$. Then we immediately have that ρ_c is the canonical pre-run of $\mathcal{A}_2$ on $\mathcal{I}$. Since ρ_c is inconsistent, it follows that $\mathcal{I} \notin L(\mathcal{A}_2)$. Hence $L(\mathcal{A}_1) \not\subseteq L(\mathcal{A}_2)$, which means that the claim is shown.

Claim 2. APTA containment coincides with APTA containment over rooted models.

The idea of the proof is very similar to the case for EA. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two APTAs over the signature $\Sigma_N \cup \Sigma_E$. First suppose that $L^r(\mathcal{A}_1) \not\subseteq L^r(\mathcal{A}_2)$. Then there is some rooted $\Sigma_N \cup \Sigma_E$-interpretation $(\mathcal{I}, d_0)$ such that $(\mathcal{I}, d_0) \in L^r(\mathcal{A}_1)$ and $(\mathcal{I}, d_0) \notin L^r(\mathcal{A}_2)$. By $(\mathcal{I}, d_0) \in L^r(\mathcal{A}_1)$ and $L^r(\mathcal{A}_1) \subseteq L(\mathcal{A}_1)$, we have $(\mathcal{I}, d_0) \in L(\mathcal{A}_1)$. Now since $(\mathcal{I}, d_0)$ is rooted and $(\mathcal{I}, d_0) \notin L^r(\mathcal{A}_2)$, we have $(\mathcal{I}, d_0) \notin L^r(\mathcal{A}_2)$, we have $(\mathcal{I}, d_0) \notin L(\mathcal{A}_2)$. Hence $(\mathcal{I}, d_0) \in L(\mathcal{A}_1) \setminus L(\mathcal{A}_2)$, which means that $L(\mathcal{A}_1) \not\subseteq L(\mathcal{A}_2)$.

For the other direction, suppose that $L(\mathcal{A}_1) \not\subseteq L(\mathcal{A}_2)$. Then there is some pointed $\Sigma_N \cup \Sigma_E$-interpretation $(\mathcal{I}, d_0)$ such that $(\mathcal{I}, d_0) \in L(\mathcal{A}_1)$ and $(\mathcal{I}, d_0) \notin L(\mathcal{A}_2)$. Let $(\mathcal{J}, d_0)$ be the pointed interpretation obtained by unravelling $(\mathcal{I}, d_0)$ using only Σ_E-edges. Obviously, $(\mathcal{J}, d_0)$ is rooted. One can now show that $(\mathcal{J}, d_0) \in L^r(\mathcal{A}_1)$ and $(\mathcal{J}, d_0) \notin L^r(\mathcal{A}_2)$. Hence $L^r(\mathcal{A}_1) \not\subseteq L^r(\mathcal{A}_2)$, which means that the claim is proved. ❑

It thus suffices to reduce EA containment over rooted models to APTA containment over rooted models. Let $\mathcal{A} = (Q, P, \Sigma_N, \Sigma_E, \delta)$ be an EA. We define an APTA $\mathcal{A}' = (Q', \Sigma_N, \Sigma_E, q'_0, \delta', \Omega')$ as follows:

$$
\begin{aligned}
Q' &= \{q'_0\} \uplus Q \uplus \{\overline{q} \mid q \in Q\} \uplus P \\
\delta'(q'_0) &= \bigwedge_{q \in Q} (q \vee \overline{q}) \wedge \bigwedge_{r \in \Sigma_E} [r] q'_0 \\
\delta'(q) &= \Big(\bigvee_{A \to q \in \delta} A \vee \bigvee_{q_1 \wedge \cdots \wedge q_n \to q \in \delta} \bigwedge_{i=1..n} q_i \vee \bigvee_{\langle r \rangle q_1 \to q \in \delta} \langle r \rangle q_1 \Big) \\
&\quad \wedge \bigwedge_{q \to p \in \delta} p \\
&\quad \text{if } \mathsf{true} \to q \notin \delta \\
\delta'(q) &= \mathsf{true} \text{ if } \mathsf{true} \to q \in \delta \\
\delta'(\overline{q}) &= \bigwedge_{A \to q \in \delta} \neg A \wedge \bigwedge_{q_1 \wedge \cdots \wedge q_n \to q \in \delta} \bigvee_{i=1..n} \overline{q}_i \wedge \bigwedge_{\langle r \rangle q_1 \to q \in \delta} [r] \overline{q}_1 \\
&\quad \text{if } \mathsf{true} \to q \notin \delta \\
\delta'(\overline{q}) &= \mathsf{false} \text{ if } \mathsf{true} \to q \in \delta \\
\delta'(p) &= \bigwedge_{p \to p_1 \in \delta} p_1 \wedge \bigwedge_{p \to \langle r \rangle p_1 \in \delta} \langle r \rangle p_1 \wedge \bigwedge_{p \to A \in \delta} A \\
&\quad \text{if } p \to \mathsf{false} \notin \delta \\
\delta'(p) &= \mathsf{false} \text{ if } p \to \mathsf{false} \in \delta \\
\Omega'(q') &= 0 \text{ for all } q' \in Q'
\end{aligned}
$$

Lemma 46. $L^r(\mathcal{A}) = L^r(\mathcal{A}')$.

Proof. $(\Rightarrow)$ Suppose $(\mathcal{I}, d_0) \in L^r(\mathcal{A})$. Then by Lemma 40, the canonical pre-run $\rho_c^{\mathcal{I}}$ of the EA $\mathcal{A}$ on $(\mathcal{I}, d_0)$ is consistent. We need to show that there is an accepting run of the APTA $\mathcal{A}'$ on $(\mathcal{I}, d_0)$. Our argument is that such a run exists in the limit of an inductive construction we will define. In order to do that we need some preliminaries.

We say that a node x of a $Q' \times \Delta^{\mathcal{I}}$-labelled tree (T, ℓ) has a *defect* if $\ell(x) = (q', d)$, for some $q' \in Q'$, and (T, ℓ) does not respect some condition of a run of $\mathcal{A}'$ on $(\mathcal{I}, d_0)$ for (q', d), e.g., if $\delta'(q') = q_1 \wedge q_2$ then x does not have two sons y_1, y_2 with $\ell(y_1) = (q_1, d)$ and $\ell(y_2) = (q_2, d)$. A $Q' \times \Delta^{\mathcal{I}}$-labelled tree is called $\rho_c^{\mathcal{I}}$*-conforming* if for all $x \in T$,

- if $\ell(x) = (q, d)$, where $q \in Q$, then $q \in \rho_c^{\mathcal{I}}(d)$;
- if $\ell(x) = (p, d)$, where $p \in P$, then there is a witness for p at d;
- if $\ell(x) = (\overline{q}, d)$, where $q \in Q$, then $q \notin \rho_c^{\mathcal{I}}(d)$.

Claim 1. There is some $\rho_c^{\mathcal{I}}$-conforming $Q' \times \Delta^{\mathcal{I}}$-labelled tree. Define the $Q' \times \Delta^{\mathcal{I}}$-labelled tree (T, ℓ) as follows:

- $T = \{x_0\}$;
- $\ell(x_0) = (q'_0, d_0)$.

By definition, $q'_0 \notin Q \cup P$. This immediately yields that (T, ℓ) is $\rho_c^{\mathcal{I}}$-conforming. Thus, the claim holds.

Claim 2. For every $\rho_c^{\mathcal{I}}$-conforming $Q' \times \Delta^{\mathcal{I}}$-labelled tree (T, ℓ) and for every $x \in T$ with a defect, there is some $\rho_c^{\mathcal{I}}$-conforming $Q' \times \Delta^{\mathcal{I}}$-labelled tree that extends (T, ℓ) and in this new tree x lacks this defect.

Let $\ell(x) = (q', d)$. We distinguish all the cases for q'.

- $q' = q'_0$. For every $q \in Q$, let $s_q = q$ if $q \in \rho_c^{\mathcal{I}}(d)$ and let $s_q = \overline{q}$ if $q \notin \rho_c^{\mathcal{I}}(d)$; and add a new son y_q of x to T with $\ell(y_q) = (s_q, d)$. Moreover, for every $e \in \Delta^{\mathcal{I}}$ be with $(d,e) \in r^{\mathcal{I}}$, where $r \in \Sigma_E$, add a new son y_e of x to T with $\ell(y_e) = (q'_0, e)$. It is easy to see that the new (T, ℓ) is $\rho_c^{\mathcal{I}}$-conforming and that x lacks defect caused by the transition $\delta'(q'_0)$.
- $q' = q$, for some $q \in Q$. If $\mathsf{true} \to q \in \delta$, then $\delta'(q) = \mathsf{true}$ and we do not have to show anything. Therefore suppose that $\mathsf{true} \to q \notin \delta$. We remind the reader that $\rho_c^{\mathcal{I}}$ is defined as the limit of a sequence of maps $\rho_{c,0}^{\mathcal{I}} \subseteq \rho_{c,1}^{\mathcal{I}} \subseteq \rho_{c,2}^{\mathcal{I}} \cdots$ from $\Delta^{\mathcal{I}}$ to 2^Q using Conditions 1 to 4 of runs. Since we assumed that $\mathsf{true} \to q \notin \delta$, $q \in \rho_c^{\mathcal{I}}(d)$ because for some $i \in \mathbb{N}$, the antecedent of Condition 2, 3, or 4 was satisfied at $\rho_{c,i}^{\mathcal{I}}$, $q \notin \rho_{c,i}^{\mathcal{I}}(d)$, and $q \in \rho_{c,i+1}^{\mathcal{I}}(d)$. Depending on the condition applied, we can easily fix the defect of the transition caused by the big disjunct in $\delta'(q)$. Here we only show it for Condition 2. Suppose $q \in \rho_c^{\mathcal{I}}(d)$ because of some $A \to q \in \delta$. Then $i = 0$ and $d \in A^{\mathcal{I}}$. Add a son y of x with $\ell(y) = (A, d)$. It is easy to see that the big disjunct in $\delta'(q)$ is satisfied.

  Now let $p \in P$ with $q \to p$. Since (T, ℓ) is $\rho_c^{\mathcal{I}}$-conforming and $\rho_c^{\mathcal{I}}$ is consistent, there is a witness for p at d; add a new son y_p of x and set $\ell(y_p) = (p, d)$. Obviously, (T, ℓ) is $\rho_c^{\mathcal{I}}$-conforming and x lacks this defect in the new tree.
- $q' = \overline{q}$, for some $q \in Q$. We have $q \notin \rho_c^{\mathcal{I}}(d)$ since (T, ℓ) is $\rho_c^{\mathcal{I}}$-conforming. Let $A \in \Sigma_N$ with $A \to q \in \delta$. We have $d \notin A^{\mathcal{I}}$ because otherwise by the definition of $\rho_c^{\mathcal{I}}$, q would be in $\rho_c^{\mathcal{I}}(d)$, which is a contradiction. This means that $\bigwedge_{A \to q \in \delta} \neg A$ is satisfied. Now let $q_1 \wedge \ldots \wedge q_n \to q \in \delta$. It must be that $q_i \notin \rho_c^{\mathcal{I}}(d)$, for some q_i because otherwise by the definition of $\rho_c^{\mathcal{I}}$, q would be in $\rho_c^{\mathcal{I}}(d)$, which is a contradiction again. Add a new son y_{q_i} of x to T and set $\ell(y_{q_i}) = (\overline{q}_i, d)$. It is not hard to see that the resulting tree is $\rho_c^{\mathcal{I}}$-conforming and the relevant conjunct of $\delta'(\overline{q})$ is satisfied. Finally, let $\langle r \rangle q_1 \to q \in \delta$. We know that there is no $e \in \Delta^{\mathcal{I}}$ with $(d, e) \in r^{\mathcal{I}}$ and $q_1 \in \rho_c^{\mathcal{I}}(e)$ because otherwise q would be in $\rho_c^{\mathcal{I}}(d)$, which is a contradiction. Therefore add a new son y_e of x to T and set $\ell(y_e) = (\overline{q}_1, e)$, for every $e \in \Delta^{\mathcal{I}}$ with $(d, e) \in r^{\mathcal{I}}$. Observe that the resulting tree is $\rho_c^{\mathcal{I}}$-conforming and the relevant conjunct of $\delta'(\overline{q})$ is satisfied.
- $q' = p$. First we observe that $p \to \mathsf{false} \notin \delta$; for otherwise, we obtain a contradiction to the fact that $\rho_c^{\mathcal{I}}$ is consistent since (T, ℓ) is $\rho_c^{\mathcal{I}}$-conforming. Now by the fact that (T, ℓ) is $\rho_c^{\mathcal{I}}$-conforming, we know that there is a witness σ for p at d. For every $p_1 \in P$ with $p \to p_1 \in \delta$, we have $p_1 \in \sigma(d)$ by definition; add a new successor y_{p_1} of x to T and set $\ell(y_{p_1}) = (p_1, d)$. It is not hard to see that the resulting tree is $\rho_c^{\mathcal{I}}$-conforming and the relevant conjunct of $\delta'(p)$ is satisfied. For every $p_1 \in P$ with $p \to \langle r \rangle p_1 \in \delta$, by the definition of σ, we have that there is some $e \in \Delta^{\mathcal{I}}$ with $(d, e) \in r^{\mathcal{I}}$ and $p_1 \in \sigma(e)$; add a new successor y_e of x to T and set $\ell(y_e) = (p_1, e)$. Again the resulting tree is $\rho_c^{\mathcal{I}}$-conforming and the relevant conjunct of $\delta'(p)$ is satisfied. Finally, since σ is a witness for p, for every $A \in \Sigma_N$ with $p \to A \in \delta$, we know that $d \in A^{\mathcal{I}}$. Hence the relevant conjunct of $\delta'(p)$ is satisfied.

We have thus shown that every kind of defect caused by a transition can be fixed. This marks the end of the proof of the claim.

Now this direction of the lemma can be shown as follows. By Claim 1, we know that there is some $\rho_c^{\mathcal{I}}$-conforming $Q' \times \Delta^{\mathcal{I}}$-labelled tree. We start with that tree and inductively fix a defect, which we know can be done by Claim 2. In the limit of this construction, the resulting tree (T, ℓ) will be defect-free. Using the defect-freeness of (T, ℓ), it is not hard to show that (T, ℓ) is a run of $\mathcal{A}'$ on $(\mathcal{I}, d_0)$.

($\Leftarrow$) Suppose $(\mathcal{I}, d_0) \in L^r(\mathcal{A}')$. Then there is a run (T, ℓ) of the APTA $\mathcal{A}'$ on $(\mathcal{I}, d_0)$. We need to show that there is some run of $\mathcal{A}$ on $(\mathcal{I}, d_0)$. By Lemma 40, it is enough to show that the canonical pre-run $\rho_c^{\mathcal{I}}$ of $\mathcal{A}$ on $(\mathcal{I}, d_0)$ is consistent. This will be a consequence of a series of claims.

Claim 3. For all $d \in \Delta^{\mathcal{I}}$ and $q \in Q$, if $q \in \rho_c^{\mathcal{I}}(d)$ then for all $x \in T$, $\ell(x) \neq (\overline{q}, d)$.

We remind the reader that $\rho_c^{\mathcal{I}}$ is defined as the limit of a sequence of maps $\rho_{c,0}^{\mathcal{I}} \subseteq \rho_{c,1}^{\mathcal{I}} \subseteq \rho_{c,2}^{\mathcal{I}} \cdots$ from $\Delta^{\mathcal{I}}$ to 2^Q. The proof is by induction on this sequence. Our inductive hypothesis is that for every $d \in \Delta^{\mathcal{I}}$ and $q \in Q$, if $q \in \rho_{c,i}^{\mathcal{I}}(d)$ then for all $x \in T$, $\ell(x) \neq (\overline{q}, d)$.

As the base case, $\rho_{c,0}^{\mathcal{I}}$ is defined as follows:

$$\begin{aligned} \rho_{c,0}^{\mathcal{I}}(d) \quad = \quad & \{q \in Q \mid \mathsf{true} \to q \in \delta\} \cup \\ & \{q \in Q \mid A \to q \in \delta \wedge d \in A^{\mathcal{I}}\}. \end{aligned}$$

Suppose for some $d \in \Delta^{\mathcal{I}}$ and $q \in Q$ that $q \in \rho_{c,0}^{\mathcal{I}}(d)$. We proceed towards contradiction so suppose that there is some $x \in T$ such that $\ell(x) = (\overline{q}, d)$. Now $q \in \rho_{c,0}^{\mathcal{I}}(d)$ either because of (i) $\mathsf{true} \to q \in \delta$ or (ii) $A \to q \in \delta$ and $d \in A^{\mathcal{I}}$. Suppose (i). By definition, $\delta'(\overline{q}) = \mathsf{false}$. But this is a contradiction since $\ell(x) = (\overline{q}, d)$. Suppose now that (ii) holds. By definition, $\neg A$ is a conjunct of $\delta'(\overline{q})$. This implies $d \notin A^{\mathcal{I}}$, which contradicts with $d \in A^{\mathcal{I}}$. Hence we can conclude that the inductive hypothesis holds for the base case.

Now, in order to construct $\rho_{c,i+1}^{\mathcal{I}}$ from $\rho_{c,i}^{\mathcal{I}}$, we first construct an extension $\sigma_{c,i}^{\mathcal{I}}$ of $\rho_{c,i}^{\mathcal{I}}$ by exhaustively applying Condition 3 of runs, and then construct $\rho_{c,i+1}^{\mathcal{I}}$ as an extension of $\sigma_{c,i}^{\mathcal{I}}$ by applying the Condition 4 of runs. Denote by $\sigma_{c,i,0}^{\mathcal{I}}, \sigma_{c,i,1}^{\mathcal{I}} \cdots$ the sequence of maps such that $\sigma_{c,i,0}^{\mathcal{I}} = \rho_{c,i}^{\mathcal{I}}$ and whose limit is $\sigma_{c,i}^{\mathcal{I}}$. By induction on this sequence, we now show that for every $d \in \Delta^{\mathcal{I}}$ and $q \in Q$, if $q \in \sigma_{c,i,j}^{\mathcal{I}}(d)$ then for all $x \in T$, $\ell(x) \neq (\overline{q}, d)$.

The base case is satisfied immediately because by assumption, $\sigma_{c,i,0}^{\mathcal{I}} = \rho_{c,i}^{\mathcal{I}}$ and $\rho_{c,i}^{\mathcal{I}}$ satisfies this property.

For the inductive step, suppose for some $q_1 \wedge \ldots \wedge q_n \to q \in \delta$ and some $d \in \Delta^{\mathcal{I}}$ that $\{q_1, \ldots, q_n\} \subseteq \sigma_{c,i,j}^{\mathcal{I}}(d)$, $q \notin \sigma_{c,i,j}^{\mathcal{I}}(d)$, and $q \in \sigma_{c,i,j+1}^{\mathcal{I}}(d)$. We proceed towards contradiction so assume that there is some $x \in T$ such that $\ell(x) = (\overline{q}, d)$. By definition, $\overline{q}_1 \vee \ldots \vee \overline{q}_n$ is a conjunct of $\delta'(\overline{q})$. This means there is some $y \in T$ such that $\ell(y) =$

$(\overline{q}_k, d)$, for some $k \in \{1, \ldots, n\}$. But this contradicts with the fact that $\sigma^{\mathcal{I}}_{c,i,j}$ satisfies the inductive hypothesis.

Thus we conclude that $\sigma_{c,i}$ satisfies the required property. Now denote by $\tau^{\mathcal{I}}_{c,i,0}, \tau^{\mathcal{I}}_{c,i,1} \cdots$ the sequence of maps such that $\tau^{\mathcal{I}}_{c,i,0} = \sigma^{\mathcal{I}}_{c,i}$ and whose limit is $\rho^{\mathcal{I}}_{c,i+1}$. By induction on this sequence, we now show that for every $d \in \Delta^{\mathcal{I}}$ and $q \in Q$, if $q \in \tau^{\mathcal{I}}_{c,i,j}(d)$ then for all $x \in T$, $\ell(x) \neq (\overline{q}, d)$.

The base case is satisfied immediately because by assumption, $\tau^{\mathcal{I}}_{c,i,0} = \sigma^{\mathcal{I}}_{c,i}$ and $\sigma^{\mathcal{I}}_{c,i}$ satisfies this property.

For the inductive step, suppose for some $\langle r\rangle q_1 \to q \in \delta$ and some $d, e \in \Delta^{\mathcal{I}}$ that $(d, e) \in r^{\mathcal{I}}$, $q_1 \in \tau^{\mathcal{I}}_{c,i,j}(e)$, $q \notin \tau^{\mathcal{I}}_{c,i,j}(d)$, and $q \in \tau^{\mathcal{I}}_{c,i,j+1}(d)$. We proceed towards contradiction so assume that there is some $x \in T$ such that $\ell(x) = (\overline{q}, d)$. By definition, $[r]\overline{q}_1$ is a conjunct of $\delta'(\overline{q})$. This means there is some $y \in T$ such that $\ell(y) = (\overline{q}_1, e)$. But this contradicts with the fact that $\tau^{\mathcal{I}}_{c,i,j}$ satisfies the inductive hypothesis.

Hence we conclude that the inductive hypothesis holds for all $\tau_{c,i,j}$ and in particular $\rho^{\mathcal{I}}_{c,i+1}$. This means that the claim is proved.

Claim 4. For all $d \in \Delta^{\mathcal{I}}$ and $q \in Q$, if $q \in \rho^{\mathcal{I}}_c(d)$ then there is some $x \in T$ such that $\ell(x) = (q, d)$.

Suppose for some $d \in \Delta^{\mathcal{I}}$ and $q \in Q$ that $q \in \rho^{\mathcal{I}}_c(d)$. Since d is reachable from d_0 along a path of Σ_E-edges, the conjunct $\bigwedge_{r \in \Sigma_E}[r]q'_0$ of $\delta'(q'_0)$ guarantees that there is some $x \in T$ such that $\ell(x) = (q'_0, d)$. Then by the conjunct $\bigwedge_{q \in Q}(q \vee \overline{q})$ of $\delta'(q'_0)$, we have that there is some $y \in T$ such that either $\ell(y) = (q, d)$ or $\ell(y) = (\overline{q}, d)$. By $q \in \rho^{\mathcal{I}}_c(d)$ and Claim 3, we know that the latter is not possible. Hence we conclude that $\ell(y) = (q, d)$, which means that the claim is shown.

Suppose $q^\star \to p^\star \in \delta$ and for some $d^\star \in \Delta^{\mathcal{I}}$, $q^\star \in \rho_c(d^\star)$. We are going to define a sequence of maps $\sigma_0 \subseteq \sigma_1 \subseteq \sigma_2 \cdots$ from $\Delta^{\mathcal{I}}$ to 2^P such that the limit in this sequence is a witness for $p^\star$ at $d^\star$. We need some preliminaries.

We say that a map $\sigma : \Delta^{\mathcal{I}} \to 2^P$ has a *defect* if for some $p \in P$ and $d \in \Delta^{\mathcal{I}}$, we have $p \in \sigma(d)$ and σ does not satisfy Condition 6, 7, 8, or 9 of runs. A map $\sigma : \Delta^{\mathcal{I}} \to 2^P$ is (T, ℓ)*-conforming* if for every $d \in \Delta^{\mathcal{I}}$ and $p \in P$ if $p \in \sigma(d)$ then there is some $x \in T$ with $\ell(x) = (p, d)$.

Claim 5. There is some (T, ℓ)-conforming map $\sigma : \Delta^{\mathcal{I}} \to 2^P$ with $p^\star \in \sigma(d^\star)$.

For all $d \in \Delta^{\mathcal{I}}$, define $\sigma(d) = \{p^\star\}$ if $d = d^\star$; and $\sigma(d) = \emptyset$ if $d \neq d^\star$. By $q^\star \in \rho_c(d^\star)$ and the Claim 4 we have that there is some $x \in T$ such that $\ell(x) = (q^\star, d^\star)$. Then by definition, $p^\star$ is a conjunct of $\delta'(q)$. This implies that there is some $y \in T$ such that $\ell(y) = (p^\star, d^\star)$. Hence σ is (T, ℓ)-conforming and $p^\star \in \sigma(d^\star)$. This marks the end of the proof of the claim.

Claim 6 For every (T, ℓ)-conforming map $\sigma : \Delta^{\mathcal{I}} \to 2^P$ with a defect, there is some (T, ℓ)-conforming map σ' that extends σ and lacks this defect.

We need to distinguish between Condition 6, 7, 8, and 9 of runs.

- Suppose $p \to p_1 \in \delta$, $p \in \sigma(d)$, and $p_1 \notin \sigma(d)$. Since σ is (T, ℓ)-conforming, there is some $x \in T$ such that $\ell(x) = (p, d)$. Then by definition, p_1 is a conjunct of $\delta'(p)$. This implies that there is some $y \in T$ such that $\ell(y) = (p_1, d)$. Add p to $\sigma(d)$ and denote by σ' the resulting function. Clearly, σ' is (T, ℓ)-conforming.

The remaining items are left as an exercise. Hence the claim follows.

Now we can show that the canonical pre-run $\rho^{\mathcal{I}}_c$ of $\mathcal{A}$ on $(\mathcal{I}, d_0)$ is consistent. We have assumed for an arbitrary $q^\star \to p^\star \in \delta$ and $d^\star \in \Delta^{\mathcal{I}}$ that $q^\star \in \rho_c(d^\star)$. By Claim 5, we know that there is some (T, ℓ)-conforming map $\sigma_0 : \Delta^{\mathcal{I}} \to 2^P$ with $p^\star \in \sigma_0(d^\star)$. We start with that map and inductively fix a defect, which we know can be done by Claim 6. In the limit of this construction, the resulting map σ is defect-free and $p^\star \in \sigma(d^\star)$. Hence σ is a witness for $p^\star$ at $d^\star$.

❏

Lemma 18. The class of languages accepted by EAs is closed under intersection. It is not closed under complementation and union.

Proof. (sketch) For intersection, let $\mathcal{A}_i = (Q_i, P_i, \Sigma_N, \Sigma_E, p_{0,i}, \delta_i)$ be an EA, for $i \in \{1, 2\}$. Assume w.l.o.g. that Q_1, Q_2, P_1, P_2 are pairwise disjoint. Define a new automaton $\mathcal{A} = (Q, P, \Sigma_N, \Sigma_E, p_0, \delta)$ by setting $Q = Q_1 \uplus Q_2$, $P = P_1 \uplus P_2 \uplus \{p_0\}$, and

$$\delta = \delta_1 \cup \delta_2 \cup \{p_0 \to p_{0,1}, p_0 \to p_{0,2}\}.$$

It is not hard to verify that $L(\mathcal{A}) = L(\mathcal{A}_1) \cap L(\mathcal{A}_2)$.

For negation, note that we can easily build an EA $\mathcal{A}$ such that $L(\mathcal{A})$ consists of all pointed interpretations that satisfy the $\mathcal{EL}$-CI $A \sqsubseteq \exists r.B$.

For complementation, it suffices to note that EAs are closed under intersection, that closure under both intersection and complementation means that containment can be reduced to emptiness, and that by Theorem 15 emptiness of EAs is strictly simpler than containment.

For union, note that we can easily build EAs $\mathcal{A}_1$ and $\mathcal{A}_2$ such that $\mathcal{A}_i$ consists of all pointed interpretations that satisfy the $\mathcal{EL}$-CI $A \sqsubseteq B_i$. Now assume there is an EA $\mathcal{A}$ with $L(\mathcal{A}) = L(\mathcal{A}_1) \cup L(\mathcal{A}_2)$. Let $\mathcal{I}_1$, $\mathcal{I}_2$, and $\mathcal{I}$ be interpretations that consist of a single point d such that d satisfies A and B_1 in $\mathcal{I}_1$, A and B_2 in $\mathcal{I}_2$, and only A in $\mathcal{I}$. We have $\mathcal{I}_1, \mathcal{I}_2 \in L(\mathcal{A})$, but $\mathcal{I} \notin L(\mathcal{A})$. As $\mathcal{I}$ is the product of $\mathcal{I}_1$ and $\mathcal{I}_2$, this is impossible by Lemma 16. ❏

D Proofs for Section 6

For the proofs of Theorems 20 and 21 we require some preparation.

We introduce

- canonical models and their properties for EAs;
- for any EA $\mathcal{A}$, an equivalence relation $\sim_{\mathcal{A}}$ between pointed interpretations;
- bisimulations and the finite depth versions of equisimulations and bisimulations.

For a given EA $\mathcal{A} = (Q, P, \Sigma_N, \Sigma_E, \delta)$ and $p, p' \in Q \cup P$, we set $p \rightarrow^* p'$ if $p = p'$ or there exist $p_0, \dots, p_n$ such that $p = p_0$, $p_n = p'$ and $p_i \rightarrow p_{i+1} \in \delta$ for all $i < n$. We set $p \rightarrow^* \langle r \rangle p'$ if there exists p_0 with $p \rightarrow^* p_0$, $p_0 \rightarrow \langle r \rangle p' \in \delta$; we set $p \rightarrow^* A$ if there exists p' with $p \rightarrow^* p'$ and $p' \rightarrow A \in \delta$.

Proposition 47 (Canonical Extension)**.** *Let $\mathcal{I}$ be an interpretation such that for every $d \in \Delta^{\mathcal{I}}$ there exists a model $\mathcal{J} \in L(\mathcal{A})$ with $(\mathcal{I}, d) \leq (\mathcal{J}, e)$ for some $e \in \Delta^{\mathcal{J}}$. Then one can construct an extension $\mathcal{I}' \in L(\mathcal{A})$ of $\mathcal{I}$ such that for every $\mathcal{J} \in L(\mathcal{A})$: from $(\mathcal{I}, d) \leq (\mathcal{J}, e)$ follows $(\mathcal{I}', d) \leq (\mathcal{J}, e)$.*

Proof. (Sketch) Assume $\mathcal{I}$ is given. First define $\mathcal{I}_0$ as follows

- $\Delta^{\mathcal{I}_0} = \Delta^{\mathcal{I}} \cup P$;
- $r^{\mathcal{I}_0} = r^{\mathcal{I}} \cup \{(p_1, p_2) \mid p_1 \rightarrow^* \langle r \rangle p_2\}$, for $r \in \mathsf{N_R}$;
- $A^{\mathcal{I}_0} = A^{\mathcal{I}} \cup \{p \in P \mid p \rightarrow^* A\}$, for $A \in \mathsf{N_C}$.

Consider the following two update rules:

- For every $A \in \Sigma_N$:

$$A^{\mathcal{J}} := A^{\mathcal{J}} \cup \{d \mid q \in \rho_c^{\mathcal{J}}(d), q \rightarrow^* A\}$$

- For every $r \in \Sigma_E$:

$$r^{\mathcal{J}} := r^{\mathcal{J}} \cup \{(d, p') \mid q \in \rho_c^{\mathcal{J}}(d), q \rightarrow^* \langle r \rangle p'\}$$

Let $\mathcal{I}_0, \mathcal{I}_1, \dots$ be the sequence of interpretations obtained by applying the two rules exhaustively to $\mathcal{I}_0$. Clearly the sequences stabilizes, say at n. Let $\mathcal{I}_n^*$ denote the subinterpretation of $\mathcal{I}_n$ generated by points reachable from $\Delta^{\mathcal{I}}$. One can show that that $\mathcal{I}' := \mathcal{I}_n^*$ is as required. ❑

By consider the interpretation $\mathcal{I}$ corresponding to an $\mathcal{EL}$-concept C, we obtain a canonical model for an EA $\mathcal{A}$ and a concept C:

Proposition 48. *Let C be a concept with $L(\mathcal{A}) \not\models C \sqsubseteq \bot$. Then one can construct an interpretation $\mathcal{I}_C$ with $d_C \in \Delta^{\mathcal{I}_C}$ and $\mathcal{I}_C \in L(\mathcal{A})$ such that for every $\mathcal{J} \in L(\mathcal{A})$ with $d \in C^{\mathcal{J}}$: $(\mathcal{I}_C, d_C) \leq (\mathcal{J}, d)$.*

Given an EA $\mathcal{A}$, we set $(\mathcal{I}_1, d_1) \sim_{\mathcal{A}} (\mathcal{I}_2, d_2)$ iff

1. $\rho_c^{\mathcal{I}_1}(d_1) = \rho_c^{\mathcal{I}_2}(d_2)$;
2. the same $p \in P$ have a witness at d_1 and d_2.

Lemma 49. *$(\mathcal{I}_1, d_1) \approx_\Sigma (\mathcal{I}_2, d_2)$ implies $(\mathcal{I}_1, d_1) \sim_{\mathcal{A}} (\mathcal{I}_2, d_2)$.*

For a tree interpretation $\mathcal{I}$ and $d \in \Delta^{\mathcal{I}}$, we denote by $\mathcal{I}(d)$ the tree interpretation generated by the subtree generated by d in $\mathcal{I}$.

Lemma 50. *Let $\mathcal{I}$ be a tree interpretation and $d \in \Delta^{\mathcal{I}}$. Assume $(\mathcal{I}(d), d) \sim_{\mathcal{A}} (\mathcal{J}, \rho^{\mathcal{J}})$ for a tree interpretation $\mathcal{J} \in L(\mathcal{A})$. Replace $\mathcal{I}(d)$ by $\mathcal{J}$ in $\mathcal{I}$ and denote the resulting tree interpretation by $\mathcal{K}$. Then $\mathcal{I} \in L(\mathcal{A})$ iff $\mathcal{K} \in L(\mathcal{A})$.*

Proof. Can be proved using canonical run. ❑

In addition to simulations and equisimulation, we require bisimulations. We characterize equivalence w.r.t. $\mathcal{ALC}_\Sigma$-concepts using bisimulations. Two pointed interpretations are *$\mathcal{ALC}_\Sigma$-m-equivalent*, in symbols $(\mathcal{I}_1, d_1) \equiv^m_{\mathcal{ALC},\Sigma} (\mathcal{I}_2, d_2)$, if, and only if, for all $\mathcal{ALC}_\Sigma$-concepts C with $\mathsf{rd}(C) \leq m$, $d_1 \in C^{\mathcal{I}_1}$ iff $d_2 \in C^{\mathcal{I}_2}$. They are *$\mathcal{EL}_\Sigma$-$m$-equivalent*, in symbols $(\mathcal{I}_1, d_1) \equiv^m_{\mathcal{EL},\Sigma} (\mathcal{I}_2, d_2)$, if, and only if, for all $\mathcal{EL}_\Sigma$-concepts C with $\mathsf{rd}(C) \leq m$, $d_1 \in C^{\mathcal{I}_1}$ iff $d_2 \in C^{\mathcal{I}_2}$.

The corresponding model-theoretic notion are m-bisimilarity and m-equisimilarity. We define m-bisimilarity inductively as follows: $(\mathcal{I}_1, d_1)$ and $(\mathcal{I}_2, d_2)$ are

- *$(\Sigma, 0)$-bisimilar*, in symbols $(\mathcal{I}_1, d_1) \sim^0_\Sigma (\mathcal{I}_2, d_2)$, if $d_1 \in A^{\mathcal{I}_1}$ iff $d_2 \in A^{\mathcal{I}_2}$ for all $A \in \Sigma \cap \mathsf{N_C}$.
- *$(\Sigma, n+1)$-bisimilar*, in symbols $(\mathcal{I}_1, d_1) \sim^{n+1}_\Sigma (\mathcal{I}_2, d_2)$, if $(\mathcal{I}_1, d_1) \sim^0_\Sigma (\mathcal{I}_2, d_2)$ and
  - for all $(d_1, e_1) \in r^{\mathcal{I}_1}$ there exists $e_2 \in \Delta^{\mathcal{I}_2}$ such that $(d_2, e_2) \in r^{\mathcal{I}_2}$ and $(\mathcal{I}_1, e_1) \sim^n_\Sigma (\mathcal{I}_2, e_2)$, for all $r \in \Sigma$;
  - for all $(d_2, e_2) \in r^{\mathcal{I}_2}$ there exists $e_1 \in \Delta^{\mathcal{I}_1}$ such that $(d_1, e_1) \in r^{\mathcal{I}_1}$ and $(\mathcal{I}_1, e_1) \sim^n_\Sigma (\mathcal{I}_2, e_2)$, for all $r \in \Sigma$.

(Σ, m)-equisimilarity, denoted by $(\mathcal{I}_1, d_1) \approx^m_\Sigma (\mathcal{I}_2, d_2)$, is defined in the obvious way. Σ-bisimilarity, denoted by $(\mathcal{I}_1, d_1) \sim_\Sigma (\mathcal{I}_2, d_2)$, is defined as usual.

Lemma 51. *(1) For all pointed interpretations $(\mathcal{I}_1, d_1)$ and $(\mathcal{I}_2, d_2)$, all signatures Σ, and all $m \geq 0$: $(\mathcal{I}_1, d_1) \equiv^m_{\mathcal{ALC},\Sigma} (\mathcal{I}_2, d_2)$ if, and only if, $(\mathcal{I}_1, d_1) \sim^m_\Sigma (\mathcal{I}_2, d_2)$.*

(2)For all pointed interpretations $(\mathcal{I}_1, d_1)$ and $(\mathcal{I}_2, d_2)$, all signatures Σ, and all $m \geq 0$: $(\mathcal{I}_1, d_1) \equiv^m_{\mathcal{EL},\Sigma} (\mathcal{I}_2, d_2)$ if, and only if, $(\mathcal{I}_1, d_1) \approx^m_\Sigma (\mathcal{I}_2, d_2)$.

The following result is proved in (Lutz, Piro, and Wolter 2011).

Lemma 52. *Let $(\mathcal{I}_1, d_1)$ and $(\mathcal{I}_2, d_2)$ be pointed interpretations such that $(\mathcal{I}_1, d_1) \sim^m_\Sigma (\mathcal{I}_2, d_2)$. Then there exist Σ-tree interpretations $\mathcal{J}_1$ and $\mathcal{J}_2$ such that*

- *$(\mathcal{J}_1, \rho^{\mathcal{J}_1}) \sim_\Sigma (\mathcal{I}_1, d_1)$;*
- *$(\mathcal{J}_2, \rho^{\mathcal{J}_2}) \sim_\Sigma (\mathcal{I}_2, d_2)$;*
- *$\mathcal{J}_1^{\leq m} = \mathcal{J}_2^{\leq m}$.*

Moreover, if $\mathcal{I}_1$ and $\mathcal{I}_2$ have finite outdegree, then one can find such $\mathcal{J}_1$ and $\mathcal{J}_2$ that have finite outdegree.

In this section, whenever we have two EAs $\mathcal{A}_1$ and $\mathcal{A}_2$, we denote by Σ the union of their alphabets.

Lemma 53. *Let $\mathcal{T}^{n-1}_\Sigma(\mathcal{A}_1) = \mathcal{T}^{n-1}_\Sigma(\mathcal{A}_2)$; that is, $L(\mathcal{A}_2)$ validate the same $\mathcal{EL}$-inclusions of role depth $\leq n-1$. Assume $(\mathcal{I}_1, d_1) \approx^n_\Sigma (\mathcal{I}_2, d_2)$ are such that $\mathcal{I}_1 \in L(\mathcal{A}_1)$ and $\mathcal{I}_2 \in L(\mathcal{A}_2)$. Then there exist $\mathcal{J}_1$, $\mathcal{J}_2$ such that*

- *$(\mathcal{J}_1, d_1) \approx^n_\Sigma (\mathcal{J}_2, d_2)$;*
- *For every $(d_1, e_1) \in r^{\mathcal{J}_1}$ there exists $(d_2, e_2) \in r^{\mathcal{J}_2}$ such that e_1 and e_2 are $(\Sigma, n-1)$-equisimilar.*
- *vice versa;*
- *$\mathcal{J}_1 \in L(\mathcal{A}_1)$ and $\mathcal{J}_2 \in L(\mathcal{A}_2)$;*
- *$(\mathcal{I}_1, d_1) \approx_\Sigma (\mathcal{J}_1, d_1)$;*
- *$(\mathcal{I}_2, d_2) \approx_\Sigma (\mathcal{J}_2, d_2)$.*

Proof. Let $(d_1, e_1) \in r^{\mathcal{I}_1}$ and assume there is no $(\Sigma, n-1)$-equisimilar r-successor of d_2 in $\mathcal{I}_2$. We find $e' \in \Delta^{\mathcal{I}_2}$ with $(d_2, e') \in r^{\mathcal{I}_2}$ and $(\mathcal{I}_1, e_1) \leq (\mathcal{I}_2, e')$. Consider the $\mathcal{EL}$-concept

$$C = \bigsqcap_{E \in X} E$$

where X is the set of $\mathcal{EL}_\Sigma$-concepts with $e_1 \in E^{\mathcal{I}_1}$ and $\mathsf{rd}(E) \leq n-1$. Take the canonical model $\mathcal{I}_C$ from Proposition 48 (with root d_C) in $L(\mathcal{A}_2)$ and hook it to d_2 by making d_C an r-successor to d_2 in $\mathcal{I}_2$. Denote the resulting interpretation by $\mathcal{I}_2'$. We show:

- $(\mathcal{I}_1, e_1) \approx_\Sigma^{n-1} (\mathcal{I}_C, d_C)$;
- $(\mathcal{I}_2, d_2) \approx_\Sigma (\mathcal{I}_2', d_2)$;
- $\mathcal{I}_2' \in L(\mathcal{A}_2)$.

Point 1 follows from the condition that $L(\mathcal{A}_1)$ and $L(\mathcal{A}_2)$ validate the same inclusions of role-depth $\leq n-1$.

For Point 2 observe that $(\mathcal{I}_2, d_2) \leq_\Sigma (\mathcal{I}_2', d_2)$ is trivial since $\mathcal{I}_2'$ is an extension of $\mathcal{I}_2$. $(\mathcal{I}_2', d_2) \leq_\Sigma (\mathcal{I}_2, d_2)$ follows from $(\mathcal{I}_C, d_C) \leq_\Sigma (\mathcal{I}_2, e')$.

Point 3 follows from Point 2 and Lemmas 49 and 50.

Now, first we do the same construction for all e with $(d_1, e) \in r^{\mathcal{I}_1}$ for which there is no $(\Sigma, n-1)$-equisimilar r-successor of d_2 in $\mathcal{I}_2$. Then we do the dual construction for all e with $(d_1, e) \in r^{\mathcal{I}_2}$ for which there is no $(\Sigma, n-1)$-equisimilar r-successor of d_1 in $\mathcal{I}_1$. The resulting interpretations are as required. ❑

As a consequence, we obtain by induction:

Lemma 54. *Assume $\mathcal{T}_\Sigma^n(\mathcal{A}_1) = \mathcal{T}_\Sigma^n(\mathcal{A}_2)$, that is $L(\mathcal{A}_1)$ and $L(\mathcal{A}_2)$ validate the same $\mathcal{EL}_\Sigma$-inclusions of role depth $\leq n$. Assume $(\mathcal{I}_1, d_1) \approx_\Sigma^n (\mathcal{I}_2, d_2)$, $\mathcal{I}_1 \in L(\mathcal{A}_1)$, and $\mathcal{I}_2 \in L(\mathcal{A}_2)$. Then there exist tree-interpretations $\mathcal{J}_1$, $\mathcal{J}_2$ of finite outdegree such that*

- $\mathcal{J}_1^{\leq n} = \mathcal{J}_2^{\leq n}$;
- $\mathcal{J}_1 \in L(\mathcal{A}_1)$ *and* $\mathcal{J}_2 \in L(\mathcal{A}_2)$;
- $(\mathcal{I}_1, d_1) \approx_\Sigma (\mathcal{J}_1, d_1)$ *and* $(\mathcal{I}_2, d_2) \approx_\Sigma (\mathcal{J}_2, d_2)$.

Proof. Using Lemma 53 one first obtains (Σ, n)-bisimilar interpretations $(\mathcal{J}_1, d_1)$ and $(\mathcal{J}_2, d_2)$ with the properties of Points 2 and 3. Using Lemma 52 one can transform them into tree interpretations. ❑

We are now in a position to prove Theorem 20 which we first formulate again.

Theorem 20 Let $\mathcal{A}$ be an EA. For all $m > 0$, the following conditions are equivalent:

1. There exists $k > m$ such that $\mathcal{T}_\Sigma^m(\mathcal{A}) \not\models \mathcal{T}_\Sigma^k(\mathcal{A})$;
2. there exist two tree interpretations, $\mathcal{I}_1$ and $\mathcal{I}_2$, of finite outdegree such that
   - $\mathcal{I}_1^{\leq m} = \mathcal{I}_2^{\leq m}$;
   - $\mathcal{I}_1 \in L(\mathcal{A})$;
   - $\mathcal{I}_2 \notin L(\mathcal{A})$;
   - For all sons d of $\rho^{\mathcal{I}_2}$: $\mathcal{I}_2(d) \in L(\mathcal{A})$.

Proof. Assume first that $\mathcal{T}_\Sigma^m(\mathcal{A}) \not\models \mathcal{T}_\Sigma^k(\mathcal{A})$ for some $k > m$. Then there exists $m' \geq m$ such that $\mathcal{T}_\Sigma^{m'}(\mathcal{A}) \not\models \mathcal{T}_\Sigma^{m'+1}(\mathcal{A})$. There exists a tree interpretation $\mathcal{I}$ of finite outdegree such that $\mathcal{I} \models \mathcal{T}_\Sigma^{m'}(\mathcal{A})$ and $\rho^{\mathcal{I}} \in C_0^{\mathcal{I}} \setminus D_0^{\mathcal{I}}$ for some $C_0 \sqsubseteq D_0 \in \mathcal{T}_\Sigma^{m'+1}(\mathcal{A})$.

Let

$$X_1 = \{E \mid C \text{ a } \mathcal{EL}_\Sigma\text{-concept}, \mathsf{rd}(E) \leq m', \rho^{\mathcal{I}} \in E^{\mathcal{I}}\}$$

and

$$X_2 = \{E \mid C \text{ a } \mathcal{EL}_\Sigma\text{-concept}, \mathsf{rd}(E) \leq m', \rho^{\mathcal{I}} \notin E^{\mathcal{I}}\}$$

Since $\mathcal{I}$ is a model of $\mathcal{T}_\Sigma^{m'}(\mathcal{A})$, $C \sqcap D$ is satisfiable in an interpretation in $L(\mathcal{A})$, where

$$C = \bigsqcap_{E \in X_1}, \quad D = \bigsqcap_{E \in X_2} \neg E$$

Thus, there exists a tree interpretation $\mathcal{J}$ of finite outdegree in $L(\mathcal{A})$ such that

$$(\mathcal{I}, \rho^{\mathcal{I}}) \approx_\Sigma^{m'} (\mathcal{J}, \rho^{\mathcal{J}}).$$

By Lemma 54 (when applying it take an EA $\mathcal{A}_1$ such that with $L(\mathcal{A}_1)$ equals the class of models of $\mathcal{T}_\Sigma^{m'}(\mathcal{A})$), we can assume:

- $\mathcal{I}^{\leq m'} = \mathcal{J}^{\leq m'}$;
- $\mathcal{J} \in L(\mathcal{A})$;
- $\mathcal{I} \models \mathcal{T}_\Sigma^{m'}(\mathcal{A})$;
- $\rho^{\mathcal{I}} \notin C_0^{\mathcal{I}} \setminus D_0^{\mathcal{I}}$;

For every son d of $\rho^{\mathcal{I}}$, as $\mathcal{I}(d)$ is a model of $\mathcal{T}_\Sigma^{m'}(\mathcal{A})$ we can argue as above and find tree interpretations $\mathcal{G}_d$ and $\mathcal{K}_d$ of finite outdegree such that

- $(\mathcal{G}_d, \rho^{\mathcal{G}_d}) \approx_\Sigma (\mathcal{I}(d), d)$;
- $\mathcal{G}_d^{\leq m'} = \mathcal{K}_d^{\leq m'}$;
- $\mathcal{K}_d \in L(\mathcal{A})$;

We have

$$(\mathcal{J}(d), d) \approx_\Sigma^{m'-1} (\mathcal{K}_d, \rho^{\mathcal{K}_d}).$$

Thus, by Lemma 54, we find tree interpretations $\mathcal{J}_d \in L(\mathcal{A})$ and $\mathcal{M}_d \in L(\mathcal{A})$ of finite outdegree such that

- $(\mathcal{J}_d, \rho^{\mathcal{J}_d}) \approx_\Sigma (\mathcal{J}(d), d)$;
- $(\mathcal{K}_d, \rho^{\mathcal{K}_d}) \approx_\Sigma (\mathcal{M}_d, \rho^{\mathcal{M}_d})$;
- $\mathcal{J}_d^{\leq m'-1} = \mathcal{M}_d^{\leq m'-1}$.

Now define $\mathcal{I}_1$ by replacing, for every son d of $\rho^{\mathcal{J}}$, $\mathcal{J}(d)$ by $\mathcal{J}_d$ in $\mathcal{J}$. Define $\mathcal{I}_2$ by replacing, for every son d of $\rho^{\mathcal{I}}$, $\mathcal{I}(d)$ by $\mathcal{M}_d$ in $\mathcal{I}$. It is readily checked that $\mathcal{I}_1$ and $\mathcal{I}_2$ are as required:

- $\mathcal{I}_1^{\leq m} = \mathcal{I}_2^{\leq m}$: since $m' \geq m$ it is sufficient to show $\mathcal{I}_1^{\leq m'} = \mathcal{I}_2^{\leq m'}$. But since $\mathcal{I}^{\leq 1} = \mathcal{J}^{\leq 1}$ this follows from $\mathcal{J}_d^{\leq m'-1} = \mathcal{M}_d^{\leq m'-1}$ for every son d of $\rho^{\mathcal{I}}$.
- $\mathcal{I}_1 \in L(\mathcal{A})$ follows, by Lemmas 49 and 50, from
  - $\mathcal{J}_d \in L(\mathcal{A})$ for all sons d of $\rho^{\mathcal{I}_1}$;

– $(\mathcal{J}_d, \rho^{\mathcal{J}_d}) \approx_\Sigma (\mathcal{J}(d), d)$ for all sons d of $\rho^{\mathcal{I}_1}$;
– $\mathcal{J} \in L(\mathcal{A})$.

- $\mathcal{I}_2 \notin L(\mathcal{A})$ follows from $\rho^{\mathcal{I}_2} \in C_0^{\mathcal{I}_2} \setminus D_0^{\mathcal{I}_2}$. This follows from
  – $(\mathcal{G}_d, \rho^{\mathcal{G}_d}) \approx_\Sigma (\mathcal{I}(d), d)$;
  – $\mathcal{G}_d^{\leq m'} = \mathcal{K}_d^{\leq m'}$;
  – $(\mathcal{K}_d, \rho^{\mathcal{K}_d}) \approx_\Sigma (\mathcal{M}_d, \rho^{\mathcal{M}_d})$;

  for all sons d of $\rho^{\mathcal{I}_2}$, $\rho^{\mathcal{I}_2} \in C_0^{\mathcal{I}} \setminus D_0^{\mathcal{I}}$, and $\mathsf{rd}(C), \mathsf{rd}(D) \leq m'$.
- For all sons d of $\rho^{\mathcal{I}_2}$: $\mathcal{I}_2(d) \in L(\mathcal{A})$. This follows from $\mathcal{I}_2(d) = \mathcal{M}_d \in L(\mathcal{A})$.

Conversely, assume that $\mathcal{I}_1$ and $\mathcal{I}_2$ satisfy Condition 2. Then $\mathcal{I}_2$ is a model of $\mathcal{T}_\Sigma^m(\mathcal{A})$. For assume this is not the case. Then there exists $d \in \Delta^{\mathcal{I}_2}$ with $d \in C^{\mathcal{I}_2} \setminus D^{\mathcal{I}_2}$ for some $C \sqsubseteq D \in \mathcal{T}_\Sigma^m(\mathcal{A})$. If $d = \rho^{\mathcal{I}_2}$, then $d \in C^{\mathcal{I}_1} \setminus D^{\mathcal{I}_1}$, by Point 1 of Condition 2. This contradicts $\mathcal{I}_1 \in L(\mathcal{A})$. If $d \neq \rho^{\mathcal{I}_2}$, then $d \in \mathcal{I}(d')$ for some son d' of $\rho^{\mathcal{I}_2}$. This contradicts Point 4.

Now assume that $\mathcal{T}_\Sigma^m(\mathcal{A}) \models \mathcal{T}_\Sigma^k(\mathcal{A})$ for all $k > m$. Let $\mathcal{A}'$ be an EA with alphabet Σ such that $L(\mathcal{A}') = \mathsf{mod}(\mathcal{T}_\Sigma^m(\mathcal{A}))$. Then $L(\mathcal{A}) = L(\mathcal{A}')$ by Lemma 56 below. But then $\mathcal{I}_2 \in L(\mathcal{A})$ which contradicts Point 3. ❑

To prove Theorem 21, we show the following lemma, where $M_\mathcal{A}$ be $2^{|Q \cup P|}$.

Lemma 55. *Let $\mathcal{A}$ be an EA. The following conditions are equivalent:*

1. *there exists $k > M_\mathcal{A}^2 + 1$ such that $\mathcal{T}_{M_\mathcal{A}^2+1} \not\models \mathcal{T}_k$;*
2. *Condition 2 from Theorem 20 holds for $m = M_\mathcal{A}^2 + 1$;*
3. *Condition 2 from Theorem 20 holds for all $m > 0$;*
4. *There does not exists an $\mathcal{EL}_\Sigma$-TBox $\mathcal{T}$ with $\mathcal{A} \equiv_\Sigma^{\mathcal{EL}} \mathcal{T}$.*

Proof. It remains to prove the implication from (2.) to (3.). Take tree interpretations $\mathcal{I}_1$ and $\mathcal{I}_2$ satisfying Condition 2 of Theorem 20 for some $m \geq M_\mathcal{A}^2 + 1$. We show that there exist tree interpretations $\mathcal{J}_1$ and $\mathcal{J}_2$ satisfying Condition 2 of Theorem 20 for $m+1$. The implication then follows by induction.

The depth of a node d in a tree interpretation is the length of the path from its root to d. Let D be the set of $d \in \Delta^{\mathcal{I}_1}$ of depth m such that $\mathcal{I}_1(d)^{\leq 1} \neq \mathcal{I}_2(d)^{\leq 1}$ (i.e., the restrictions of $\mathcal{I}_1$ to $\{d' \mid d' \text{ son of } d \text{ in } \mathcal{I}_1\}$ and $\mathcal{I}_2$ to $\{d' \mid d' \text{ son of } d \text{ in } \mathcal{I}_2\}$ do not coincide). If $D = \emptyset$, then $\mathcal{I}_1^{\leq m+1} = \mathcal{I}_2^{\leq m+1}$ and the claim is proved. Otherwise choose $f \in D$ and consider the path

$$\rho^{\mathcal{I}_1} = d_0 r_0 d_1 \cdots r_{m-1} d_m = f$$

with $(d_i, d_{i+1}) \in r_i^{\mathcal{I}_1}$ for all $i < m$. As $m \geq M_\mathcal{A}^2 + 1$, there exists $0 < i < j \leq m$ such that both,

$$(\mathcal{I}_1, d_i) \sim_\mathcal{A} (\mathcal{I}_1, d_j), \quad (\mathcal{I}_2, d_i) \sim_\mathcal{A} (\mathcal{I}_2, d_j).$$

Replace $\mathcal{I}_1(d_j)$ by $\mathcal{I}_1(d_i)$ in $\mathcal{I}_1$ and denote the resulting interpretation by $\mathcal{K}_1$. Similarly, replace $\mathcal{I}_2(d_j)$ by $\mathcal{I}_2(d_i)$ in $\mathcal{I}_2$ and denote the resulting interpretation $\mathcal{K}_2$. By Lemma 50, $\mathcal{K}_1$ and $\mathcal{K}_2$ still have Properties (1)-(4). of Condition 2 of Theorem 20. Moreover, the set D' of all $d \in \Delta^{\mathcal{I}_1'}$ of depth m such that $\mathcal{K}_1(d)^{\leq 1} \neq \mathcal{K}_2(d)^{\leq 1}$ is a subset of D not containing f. Thus, we can proceed with D' in the same way as above until the set is empty. Denote the resulting interpretations by $\mathcal{J}_1$ and $\mathcal{J}_2$, respectively. They still have Properties (1)–(4), but now for some $m' > m$. ❑

Proposition 23, as stated in Section 6, is an immediate consequence of the following lemma. In the lemma, $L^f(\mathcal{A})$ is $L(\mathcal{A})$ restricted to interpretations with finite outdegree.

Lemma 56. *Let $\mathcal{A}_1, \mathcal{A}_2$ be EAs over the same alphabets Σ. Then the following are equivalent:*

1. *$\mathcal{T}(\mathcal{A}_2) \subseteq \mathcal{T}(\mathcal{A}_1)$;*
2. *$\mathcal{T}_\Sigma(\mathcal{A}_2) \subseteq \mathcal{T}_\Sigma(\mathcal{A}_1)$;*
3. *$L^f(\mathcal{A}_1) \subseteq L^f(\mathcal{A}_2)$.*
4. *$L(\mathcal{A}_1) \subseteq L(\mathcal{A}_2)$;*

Proof. The implication "1 $\Rightarrow$ 2" is trivial, and "2 $\Rightarrow$ 3" and "3 $\Rightarrow$ 4"are proved in Lemma 57 and Lemma 58 below, respectively. It thus remains to show "4 $\Rightarrow$ 1". Let $\mathcal{A}_1, \mathcal{A}_2$ be EAs over the same alphabet $\Sigma = \Sigma_N \cup \Sigma_E$ such that $L(\mathcal{A}_1) \subseteq L(\mathcal{A}_2)$ and let $\mathcal{A}_1 \not\models C \sqsubseteq D$. Then there is an interpretation $\mathcal{I} \in L(\mathcal{A}_1)$ such that $C^\mathcal{I} \setminus D^\mathcal{I} \neq \emptyset$. By $L(\mathcal{A}_1) \subseteq L(\mathcal{A}_2)$, we have $\mathcal{I} \in \mathcal{L}(\mathcal{A}_2)$ and consequently $\mathcal{A}_2 \not\models C \sqsubseteq D$ as required. ❑

Lemma 57. *For all EAs $\mathcal{A}_1, \mathcal{A}_2$ over the same alphabet $\Sigma = \Sigma_N \cup \Sigma_E$, $\mathcal{T}_\Sigma(\mathcal{A}_2) \subseteq \mathcal{T}_\Sigma(\mathcal{A}_1)$ implies $L^f(\mathcal{A}_1) \subseteq L^f(\mathcal{A}_2)$.*

Proof. Let $\mathcal{A}_1, \mathcal{A}_2$ be EAs over the same alphabet $\Sigma = \Sigma_N \cup \Sigma_E$ such that $\mathcal{T}_\Sigma(\mathcal{A}_2) \subseteq \mathcal{T}_\Sigma(\mathcal{A}_1)$. We start by showing a series of claims that we need for the proof.

Associate a single concept $C_{p,n}$ to each state $p \in P_2$ and each $n \geq 0$:

- $C_{p,0} = \bigsqcap\{A \mid p \to_2^* A\} \sqcap \begin{cases} \bot & \text{if } p \to_2^* \mathsf{false} \\ \top & \text{otherwise} \end{cases}$
- $C_{p,i+1} = C_{p,i} \sqcap \bigsqcap\{\exists r.C_{p',i} \mid p \to_2^* \langle r \rangle p'\}$

Using the definition of the concepts $C_{p,n}$, it is easy to prove the following.

Claim 1.

1. For all $p \in P_2$ and $m \leq n$, $\mathcal{A}_2 \models C_{p,n} \sqsubseteq C_{p,m}$.
2. If $p \to p' \in \delta_2$, then $\mathcal{A}_2 \models C_{p,n} \sqsubseteq C_{p',n}$ for all $n \geq 0$.

The proof of the claim is not hard. For Item 1, we observe that for all $i \geq 0$, we have $\mathcal{A}_2 \models C_{p,i+1} \sqsubseteq C_{p,i}$ because $C_{p,i}$ appears as a conjunct in $C_{p,i+1}$. Once this is established, Item 1 follows trivially.

For Item 2, we use induction. Suppose $p \to p' \in \delta_2$. As the base case, we need to show that $\mathcal{A}_2 \models C_{p,0} \sqsubseteq C_{p',0}$. We distinguish cases for p and p'.

- $p \to_2^* \mathsf{false}$. In this case, $\models C_{p,0} \equiv \bot$. Then it trivially follows that $\mathcal{A}_2 \models C_{p,0} \sqsubseteq C_{p',0}$.
- $p' \to_2^* \mathsf{false}$. In this case, $\models C_{p',0} \equiv \bot$, and by $p \to p'$, $\models C_{p,0} \equiv \bot$. Hence $\mathcal{A}_2 \models C_{p,0} \sqsubseteq C_{p',0}$.

- Neither $p \to_2^* \mathsf{false}$ nor $p' \to_2^* \mathsf{false}$. Let $S_p = \{A \mid p \to_2^* A\}$ and $S_{p'} = \{A \mid p' \to_2^* A\}$. Since $p \to p' \in \delta_2$, $S_p \supseteq S_{p'}$. Hence $\mathcal{A}_2 \models C_{p,0} \sqsubseteq C_{p',0}$.

For the inductive step, let $S_p = \{\exists r.C_{p_1,i} \mid p \to_2^* \langle r\rangle p_1\}$ and $S_{p'} = \{\exists r.C_{p_1,i} \mid p' \to_2^* \langle r\rangle p_1\}$. By $p \to p'$, we have $S_p \supseteq S_{p'}$; and by the inductive hypothesis, $\mathcal{A}_2 \models C_{p,i} \sqsubseteq C_{p',i}$. But then $\mathcal{A}_2 \models C_{p,i} \sqcap \bigsqcap S_p \sqsubseteq C_{p',i} \sqcap \bigsqcap S_{p'}$, i.e., $\mathcal{A}_2 \models C_{p,i+1} \sqsubseteq C_{p',i+1}$.

We have just shown that Claim 1 holds.

Claim 2. For all interpretations $\mathcal{I}$, $d \in \Delta^{\mathcal{I}}$, $p \in P_2$, and $n \geq 0$, if there is a witness for p at d then $d \in C_{p,n}^{\mathcal{I}}$.

The proof is by induction on n. As the base case, we need to show that $d \in C_{p,0}^{\mathcal{I}}$. To this aim, we will show that $d \in C^{\mathcal{I}}$, for every conjunct C of $C_{p,0}$. Let C be a conjunct of $C_{p,0}$. $C \neq \bot$ because this means $p \to_2^* \mathsf{false}$, which contradicts with the fact that there is a witness for p at d. If $C = \top$, then it immediately follows that $d \in \top^{\mathcal{I}}$. Otherwise, $C = A$, for some concept name A with $p \to_2^* A$. Since the witness is closed under Conditions 6 and 8 of runs, we have $d \in A^{\mathcal{I}}$. Hence we conclude that $d \in C_{p,0}^{\mathcal{I}}$.

For the inductive step, we show that $d \in C^{\mathcal{I}}$, for every conjunct C of $C_{p,i+1}$. If $C = C_{p,i}$, then $d \in C_{p,i}^{\mathcal{I}}$, by the inductive hypothesis. Otherwise $C = \exists r.C_{p',i}$, for some $p \to_2^* \langle r\rangle p'$. Since there is a witness σ for p at d, by Condition 6 and 7 of runs, there is some $e \in \Delta^{\mathcal{I}}$ with $(d,e) \in r^{\mathcal{I}}$ and $p' \in \sigma(e)$. By the inductive hypothesis, $e \in C_{p',i}^{\mathcal{I}}$. But then $d \in (\exists r.C_{p',i})^{\mathcal{I}}$. Hence the claim follows.

We also need to define concepts for states in Q_2 and $n \geq 0$. But first we need some definitions. For each set $W \subseteq Q_2$ and $q \in Q_2$, we write $W \to^{\wedge} q$ iff $q \in W^{\wedge}$, where $W^{\wedge}$ is obtained from W by exhaustively applying Conditions 3 of runs as a rule.

We associate with every state $q \in Q$ and every $n \geq 0$ a set $S_{q,n}$ of $\mathcal{EL}$-concepts of role depth n:

- $S_{q,0} = \{A \mid A \to q \in \delta_2\} \cup \{\top \mid \mathsf{true} \to q \in \delta_2\}$
- $S_{q,i+1}$ consists of
  - all minimal conjunctions $C_1 \sqcap \cdots \sqcap C_k$ such that $W \to^{\wedge} q$, where $W = \{q' \mid C_j \in S_{q',i}\}$;
  - all concepts $\exists r.C$ such that there is some $q' \in Q_2$ with $C \in S_{q',i}$ and $\langle r\rangle q' \to q \in \delta_2$.

Claim 3 and 4 relate the sets $S_{q,n}$ to canonical pre-runs of $\mathcal{A}_2$.

Claim 3. For all interpretations $\mathcal{I}$, $d \in \Delta^{\mathcal{I}}$, $q \in Q_2$, and $n \geq 0$, if $q \in \rho_{c,n}^{\mathcal{I},2}(d)$ then there is some $C \in S_{q,n}$ such that $d \in C^{\mathcal{I}}$.

The proof is by induction. As the base case, let $q \in \rho_{c,0}^{\mathcal{I},2}(d)$. Then by the definition of $\rho_{c,0}^{\mathcal{I},2}$, either (i) $\mathsf{true} \to q \in \delta_2$ or (ii) $d \in A^{\mathcal{I}}$ and $A \to q \in \delta_2$. Suppose (i) holds. Then by the definition of $S_{q,0}$, we obtain $\top \in S_{q,0}$. Obviously $d \in \top^{\mathcal{I}}$. Suppose now (ii) holds. Then by the definition of $S_{q,0}$, we immediately obtain $A \in S_{q,0}$. Hence in both cases, there is some $C \in S_{q,0}$ such that $d \in C^{\mathcal{I}}$.

For the inductive step, let $q \in \rho_{c,i+1}^{\mathcal{I},2}(d)$. Now we have that either $q \in \rho_{c,i}^{\mathcal{I},2}(d)$ or $q \notin \rho_{c,i}^{\mathcal{I},2}(d)$. Suppose first $q \in \rho_{c,i}^{\mathcal{I},2}(d)$. Then by the inductive hypothesis, there is some $C \in S_{q,i}$ such that $d \in C^{\mathcal{I}}$. Since $\{q\} \to^{\wedge} q$, we have by the definition of $S_{q,i+1}$ that $C \in S_{q,i+1}$. Hence the claim follows in this case. Now suppose $q \notin \rho_{c,i}^{\mathcal{I},2}(d)$. In order to construct $\rho_{c,i+1}^{\mathcal{I},2}$ from $\rho_{c,i}^{\mathcal{I},2}$, we first construct an extension $\sigma_{c,i}^{\mathcal{I},2}$ of $\rho_{c,i}^{\mathcal{I},2}$ by exhaustively applying Condition 3 of runs, and then construct $\rho_{c,i+1}^{\mathcal{I},2}$ as an extension of $\sigma_{c,i}^{\mathcal{I},2}$ by applying the Condition 4 of runs. We distinguish these two cases.

1. $q \in \sigma_{c,i}^{\mathcal{I},2}(d) \setminus \rho_{c,i}^{\mathcal{I},2}(d)$. This means $\rho_{c,i}^{\mathcal{I},2}(d) \to^{\wedge} q$, because otherwise $q \notin \sigma_{c,i}^{\mathcal{I},2}(d)$. Let $\rho_{c,i}^{\mathcal{I},2}(d) = \{q_1, \ldots, q_m\}$. By the inductive hypothesis, for every $j \in \{1, \ldots, m\}$, there is some $C_j \in S_{q_j,i}$ such that $d \in C_j^{\mathcal{I}}$. By $\{q_1, \ldots, q_m\} \to^{\wedge} q$ and the definition of $S_{q,i+1}$, there is some concept $D \in S_{q,i+1}$ that is equivalent to $C_1 \sqcap \ldots \sqcap C_m$. Then by $d \in (C_1 \sqcap \ldots \sqcap C_m)^{\mathcal{I}}$, we obtain $d \in D^{\mathcal{I}}$.
2. $q \in \rho_{c,i+1}^{\mathcal{I},2}(d) \setminus \sigma_{c,i}^{\mathcal{I}}(d)$. This means for some $e \in \Delta^{\mathcal{I}}$, $q' \in Q_2$, and $r \in \Sigma_E$, we have $(d,e) \in r^{\mathcal{I}}$, $\langle r\rangle q' \to q \in \delta_2$, and $q' \in \sigma_{c,i}(e)$. By the inductive hypothesis, there is some $C \in S_{q',i}$ such that $e \in C^{\mathcal{I}}$. By $C \in S_{q',i}$, $\langle r\rangle q' \to q \in \delta_2$, and the definition of $S_{q,i+1}$, we have $\exists r.C \in S_{q,i+1}$. Moreover, by $(d,e) \in r^{\mathcal{I}}$ and $e \in C^{\mathcal{I}}$, we have $d \in (\exists r.C)^{\mathcal{I}}$.

Hence in both cases, there is some $C \in S_{q,i+1}$ such that $d \in C^{\mathcal{I}}$. This marks the end of the proof of the Claim 3.

Claim 4. For all interpretations $\mathcal{I}$, $q \in Q_2$, $n \geq 0$, $C \in S_{q,n}$, and $d \in \Delta^{\mathcal{I}}$, if $d \in C^{\mathcal{I}}$ then $q \in \rho_{c,n}^{\mathcal{I},2}(d)$.

Let $\mathcal{I}$ be an interpretation. The proof is by induction on n. As the base case, suppose $C \in S_{q,0}$ and $d \in C^{\mathcal{I}}$. By the definition of $S_{q,0}$, we have that $C = \top$ or for some $A \to q \in \delta_2$, $C = A$. If the former holds, then trivially it holds that $d \in \top^{\mathcal{I}}$; therefore, suppose that latter holds. Then by the definition of $\rho_{c,0}^{\mathcal{I}}$, $q \in \rho_{c,0}^{\mathcal{I}}(d)$. Hence the base case satisfies the claim.

For the inductive step, suppose that $C \in S_{q,i+1}$ and $d \in C^{\mathcal{I}}$. We distinguish cases for C.

- $C = C_1 \sqcap \ldots \sqcap C_k$. Then there is some $W = \{q_1, \ldots, q_k\}$ such that $C_j \in S_{q_j,i}$ for all $j \in \{1, \ldots, k\}$ and $W \to^{\wedge} q$. By the inductive hypothesis and the fact that $d \in C_j^{\mathcal{I}}$, for all $j \in \{1, \ldots, k\}$, we obtain that $\{q_1, \ldots, q_k\} \subseteq \rho_{c,i}^{\mathcal{I}}(d)$. Since $W \to^{\wedge} q$ and we exhaustively apply Condition 3 of runs to $\rho_{c,i}^{\mathcal{I}}$ to obtain $\rho_{c,i+1}^{\mathcal{I}}$, it follows that $q \in \rho_{c,i+1}^{\mathcal{I}}(d)$.
- $C = \exists r.D$. Then there is some $q' \in Q_2$ with $D \in S_{q',i}$ and $\langle r_i\rangle q' \to q \in \delta_2$. Since $d \in (\exists r.D)^{\mathcal{I}}$, there is some $e \in \Delta^{\mathcal{I}}$ such that $(d,e) \in r^{\mathcal{I}}$ and $e \in D^{\mathcal{I}}$. By the inductive hypothesis, $q' \in \rho_{c,i}^{\mathcal{I}}(e)$. But then by the definition of $\rho_{c,i+1}^{\mathcal{I}}$, we have $q \in \rho_{c,i+1}^{\mathcal{I}}(d)$.

Hence in both cases, it follows that $q \in \rho_{c,i+1}^{\mathcal{I}}(d)$.This marks the end of the proof of Claim 3.

Claim 5 relates sets $S_{q,n}$ to concepts $C_{p,n}$.

Claim 5. For all $q \to p \in \delta_2$, if $C \in S_{q,n}$, where $n \geq 0$, then $\mathcal{A}_2 \models C \sqsubseteq C_{p,i}$, for all $i \geq 0$.

Suppose $q \to p \in \delta_2$ and $C \in S_{q,n}$, for an $n \geq 0$. Let $\mathcal{I} \in L(\mathcal{A}_2)$ and let $d \in \Delta^{\mathcal{I}}$ with $d \in C^{\mathcal{I}}$. By Claim 4, $q \in \rho^{\mathcal{I},2}_{c,n}(d)$. This means, $q \in \rho^{\mathcal{I},2}_{c}(d)$, where $\rho^{\mathcal{I},2}_{c}$ is the canonical pre-run of $\mathcal{A}_2$ on $\mathcal{I}$. Since $\mathcal{I} \in L(\mathcal{A}_2)$, $\rho^{\mathcal{I},2}_{c}$ is consistent. In particular, there is a witness for p at d. Then by Claim 2, for all $i \geq 0$, $d \in C^{\mathcal{I}}_{p,i}$. This marks the end of the proof of Claim 5.

Now we progress with the proof of the lemma by making use of the claims we have shown. Let $\mathcal{I} \in L^f(\mathcal{A}_1)$ and let $\rho^{\mathcal{I},2}_{\mathsf{c}}$ be the canonical pre-run of $\mathcal{A}_2$ on $\mathcal{I}$. We need to show that $\rho^{\mathcal{I},2}_{\mathsf{c}}$ is consistent, i.e., for each $d_0 \in \Delta^{\mathcal{I}}$, $q_0 \in \rho^{\mathcal{I},2}_{\mathsf{c}}(d_0)$, and $q_0 \to p_0 \in \delta_2$, there is a witness for p_0 at d_0. Fix such d_0, q_0, and p_0.

By the definition of $\rho^{\mathcal{I},2}_{\mathsf{c}}$, there is an $n \geq 0$ such that $q_0 \in \rho^{\mathcal{I},2}_{\mathsf{c},n}(d_0)$. By Claim 3 this implies that there is some concept $C \in S_{q_0,n}$ such that $d_0 \in C^{\mathcal{I}}$. Then by Claim 5, we have $\mathcal{A}_2 \models C \sqsubseteq C_{p_0,j}$, for all $j \geq 0$. This and $\mathcal{T}_\Sigma(\mathcal{A}_2) \subseteq \mathcal{T}_\Sigma(\mathcal{A}_1)$ imply that $\mathcal{A}_1 \models C \sqsubseteq C_{p_0,j}$, for all $j \geq 0$. Finally, by this, $\mathcal{I} \in L^f(\mathcal{A}_1)$, and $d_0 \in C^{\mathcal{I}}$, we have $d_0 \in C^{\mathcal{I}}_{p_0,j}$, for all $j \geq 0$. We use this fact to identify the desired witness for p_0 as the limit σ of a sequence $\sigma_0, \sigma_1, \ldots$ of maps from $\Delta^{\mathcal{I}}$ to 2^{P_2} such that the following condition is satisfied:

($*$) if $\sigma_i(d)$ is defined and $p \in \sigma_i(d)$, then $d \in C^{\mathcal{I}}_{p,n}$ for all $n \geq 0$.

Start with setting

$$\sigma_0(d_0) = \{p \mid p_0 \to^*_2 p\}$$

and $\sigma_0(d) = \emptyset$ for all $d \neq d_0$. Now let $p \in \sigma_0(d_0)$. By definition, $p_0 \to^*_2 p$. If $p = p_0$ then by the fact that $d_0 \in C^{\mathcal{I}}_{p_0,n}$ for all $n \geq 0$, ($*$) is immediately satisfied. Now let $p \neq p_0$ and $n \geq 0$. Then by the fact that $d_0 \in C^{\mathcal{I}}_{p_0,n}$ for all $n \geq 0$ and Point 2 of Claim 1, $d_0 \in C^{\mathcal{I}}_{p,n}$. Hence ($*$) is also satisfied in this case.

Now, σ_{i+1} is obtained by extending σ_i as follows. Whenever $p \in \sigma_i(d)$ and $p \to \langle r\rangle p' \in \delta_2$, but there is no $(d,d') \in r^{\mathcal{I}}$ such that $p' \in \sigma_i(d')$, then do the following. By ($*$) and definition of the concepts $C_{p,n}$, we have $d \in (\exists r.C_{p',n})^{\mathcal{I}}$ for all $n \geq 0$. Since $\mathcal{I}$ is of finite outdegree, there is a $(d,d') \in r^{\mathcal{I}}$ with $d' \in C_{p',n}$ for infinitely many n. By Point 1 of Claim 1, this actually yields $d' \in C_{p',n}$ for all $n \geq 0$. Add

$$\{p'' \mid p' \to^*_2 p''\}$$

to $\sigma_{i+1}(d')$. By Point 2 of Claim 1, it is clear that ($*$) is satisfied.

By definition, σ is a witness for p_0 at d_0. Hence $\mathcal{I} \in L^f(\mathcal{A}_2)$. ❏

Lemma 58. *For all EAs $\mathcal{A}_1, \mathcal{A}_2$ over the same alphabets Σ_N, Σ_E, $L^f(\mathcal{A}_1) \subseteq L^f(\mathcal{A}_2)$ implies $L(\mathcal{A}_1) \subseteq L(\mathcal{A}_2)$.*

Proof. Suppose $L(\mathcal{A}_1) \not\subseteq L(\mathcal{A}_2)$, i.e., there is an interpretation $\mathcal{I} \in L(\mathcal{A}_1) \setminus L(\mathcal{A}_2)$. We show how to restrict $\mathcal{I}$ to an intepretation $\mathcal{I}'$ of finite outdegree such that still $\mathcal{I}' \in L(\mathcal{A}_1) \setminus L(\mathcal{A}_2)$. Let $\rho^{\mathcal{I},i}_{\mathsf{c}}$ be canonical pre-run of $\mathcal{A}_i$ on $\mathcal{I}$. Since $\mathcal{I} \notin L(\mathcal{A}_2)$, there are $d_0 \in \Delta^{\mathcal{I}}$, $q_0 \in \rho^{\mathcal{I},2}_{c}$, and $q_0 \to p_0 \in \delta_2$ such that there is no witness for p_0 at d_0. For each $d \in \Delta^{\mathcal{I}}$, $q \in \rho^{\mathcal{I},1}_{c}$, and $q \to p \in \delta_1$, select a witness $\sigma_{d,p}$ for p at d. Let S be the union of all these witnesses, i.e.,

$$S(d) = \bigcup_{d' \in \Delta^{\mathcal{I}}, p \in P_1} \sigma_{d',p}(d).$$

We select a subset $W \subseteq \Delta^{\mathcal{I}}$ as the limit of a sequence $W_0 \subseteq W_1 \subseteq \cdots \subseteq \Delta^{\mathcal{I}}$. Start with setting $W_0 = \{d_0\}$ and define W_{i+1} as W_i, extended as follows:

1. if $d \in W_i$, $q \in \rho^{\mathcal{I},i}(d)$ for $i \in \{1,2\}$, $\langle r\rangle q' \to q \in \delta_i$, there is a $d' \in \Delta^{\mathcal{I}}$ with $(d,d') \in r^{\mathcal{I}}$ and $q' \in \rho^{\mathcal{I},i}(d')$, and there is no such d' in W_i, then select a single such d' and add it to W_{i+1};
2. if $d \in W_i$, $p \in S(d)$, and $p \to \langle r\rangle p' \in \delta_1$, and there is no $d' \in W_i$ such that $r(d,d') \in \mathcal{I}$ and $p' \in S(d)$, then select such a d' (which exists by definition of witnesses) and add it to W_{i+1}.

Let $\mathcal{I}'$ be the restriction of $\mathcal{I}$ to W. It is easy to see that $\mathcal{I}'$ is of finite outdegree since for each $d \in \Delta^{\mathcal{I}}$, there can be at most one successor for each $q \in Q_1 \uplus Q_2 \uplus P_1$ and $r \in \Sigma_E \cap \mathsf{N_R}$. It thus remains to show that $\mathcal{I}' \in L(\mathcal{A}_1)$ and $\mathcal{I}' \notin L(\mathcal{A}_2)$. The following is easy to prove by induction on the construction of $\rho^{\mathcal{I},i}_{\mathsf{c}}$ and $\rho^{\mathcal{I}',i}_{\mathsf{c}}$:

Claim 1. For all $d \in \Delta^{\mathcal{I}'}$ and $i \in \{1,2\}$, we have $\rho^{\mathcal{I},i}_{\mathsf{c}}(d) = \rho^{\mathcal{I}',i}_{\mathsf{c}}(d)$.

Let S' be the restriction of S to W. By construction of W and using Claim 1, it is easy to show that for every $d \in \Delta^{\mathcal{I}'}$, $q \in \rho^{\mathcal{I}',1}_{\mathsf{c}}(d)$, and $q \to p \in \delta_1$, S' is a witness for p at d. By Lemma 40, we have thus shown that $\mathcal{I}' \in L(\mathcal{A}_1)$.

Assume to the contrary of what remains to be shown that $\mathcal{I}' \in L(\mathcal{A}_2)$. Then there is a witness σ for p_0 at d_0. Using the construction of $\mathcal{I}'$, it is not hard to verify that σ is also a witness for p_0 at d_0 in $\mathcal{I}$, contradicting the fact that no such witness exists. ❏

E Proofs for Section 7

Theorem 24. Let $\mathcal{T}$ be an $\mathcal{EL}$-TBox and $\Sigma \subseteq \mathsf{sig}(\mathcal{T})$ a signature. Then there exists an EA $\mathcal{A}_{\mathcal{T},\Sigma} = (Q, P, \Sigma \cap \mathsf{N_C}, \Sigma \cap \mathsf{N_R}, \delta)$ with $|Q| \in \mathcal{O}(|\mathcal{T}|)$ such that $L(\mathcal{A}_{\mathcal{T},\Sigma}) = \mathsf{cl}^{\Sigma}_{\approx}(\mathsf{mod}(\mathcal{T}))$.

"$\subseteq$". Let $\mathcal{I} \in L(\mathcal{A}_{\mathcal{T},\Sigma})$ and ρ be a run of $\mathcal{A}_{\mathcal{T},\Sigma}$ on $\mathcal{I}$. We construct a model $\mathcal{J}$ of $\mathcal{T}$ such that for all $d \in \Delta^{\mathcal{I}}$, there is an $e \in \Delta^{\mathcal{J}}$ with $(\mathcal{I}, d) \approx_\Sigma (\mathcal{J}, e)$.

Fix, for each $D = \exists r.C \in \mathsf{sub}(\mathcal{T})$ with $r \notin \Sigma$ and each $d \in \Delta^{\mathcal{I}}$ with $p_D \in \rho(d)$ a least tree model $\mathcal{J}_{C,d}$ of C and $\mathcal{T}$, i.e., $\mathcal{J}_{C,d}$ is a tree model of $\mathcal{T}$ with root $d \in C^{\mathcal{J}_{C,d}}$ and for all models $\mathcal{J}$ of $\mathcal{T}$ and $e \in C^{\mathcal{J}}$, we have $(\mathcal{J}_{C,d}, d) \leq (\mathcal{J}, e)$.Moreover, let $\mathcal{J}_{D,d}$ be obtained from $\mathcal{J}_{C,d}$ by adding

d as a fresh root which has an r-edge into the former root. Assume w.l.o.g. that the domains of all chosen models are pairwise disjoint, and that each model $\mathcal{J}_{D,d}$ shares with $\mathcal{I}$ only the domain element d. Let Γ be the set of all models $\mathcal{J}_{D,d}$ chosen. Now define $\mathcal{J}$ as:

$$\begin{aligned} \Delta^{\mathcal{J}} &= \Delta^{\mathcal{I}} \cup \bigcup_{\mathcal{J}_{D,d} \in \Gamma} \Delta^{\mathcal{J}_{D,d}} \\ A^{\mathcal{J}} &= A^{\mathcal{I}} \cup \bigcup_{\mathcal{J}_{D,d} \in \Gamma} A^{\mathcal{J}_{D,d}} \\ r^{\mathcal{J}} &= r^{\mathcal{I}} \cup \bigcup_{\mathcal{J}_{D,d} \in \Gamma} r^{\mathcal{J}_{D,d}} \end{aligned}$$

The following claim clarifies the concept memberships of elements from $\Delta^{\mathcal{J}} \setminus \Delta^{\mathcal{I}}$. It is trivial to prove by induction on the structure of C, exploiting the fact that all interpretations in Γ are tree shaped.

Claim 3. For all $\mathcal{J}_{D,d} \in \Gamma$, $e \in \Delta^{\mathcal{J}_{D,d}} \setminus \{d\}$, and $C \in \mathsf{sub}(\mathcal{T})$, we have $e \in C^{\mathcal{J}_{D,d}}$ iff $e \in C^{\mathcal{J}}$.

The following claim clarifies the concept memberships of the elements from $\Delta^{\mathcal{I}}$.

Claim 4. For all $d \in \Delta^{\mathcal{I}}$ and $C \in \mathsf{sub}(\mathcal{T})$, the following are true:

1. $d \in C^{\mathcal{J}}$ implies that $q_C \in \rho(d)$ or $C = \exists r.D$ for some $r \notin \Sigma$ and $D \in \mathsf{sub}(\mathcal{T})$;
2. $p_C \in \rho(d)$ implies $d \in C^{\mathcal{J}}$.

Proof of claim. The proof is by induction on the structure of C:

- $C = \top$. Point 1 is easy by the transition $\mathsf{true} \rightarrow q_\top$ and Point 2 since $\top^{\mathcal{J}} = \Delta^{\mathcal{J}}$.
- $C = A \in \mathsf{N_C}$. For Point 1, $d \in A^{\mathcal{J}}$ implies $d \in A^{\mathcal{I}}$ by definition of $\mathcal{J}$, and thus the transition $A \rightarrow q_A$ yields $q_A \in \rho(d)$. For Point 2, $p_A \in \rho(d)$ implies $d \in A^{\mathcal{I}}$ due to the transition $p_A \rightarrow A$, and thus $d \in A^{\mathcal{J}}$ by definition of $A^{\mathcal{J}}$.
- $C = C_1 \sqcap C_2$. Easy using the semantics, IH, and the transitions $q_{C_1} \wedge q_{C_2} \rightarrow q_C$, $p_C \rightarrow p_{C_1}$, and $p_C \rightarrow p_{C_2}$.
- $C = \exists r.C_1$. For Point 1, let $(d,e) \in r^{\mathcal{J}}$ with $r \in \Sigma$ and $e \in C_1^{\mathcal{J}}$. By construction of $\mathcal{J}$, this implies $e \in \Delta^{\mathcal{I}}$. Thus IH yields $q_{C_1} \in \rho(e)$. Due to the transition $\langle r \rangle q_{C_1} \rightarrow q_{\exists r.C_1}$, we have $q_{\exists r.C_1} \in \rho(d)$.

  For Point 2, assume that $p_{\exists r.C_1} \in \rho(d)$. First assume that $r \in \Sigma$. Because of the transition $p_{\exists r.C} \rightarrow \langle r \rangle C$, there is a $(d,e) \in r^{\mathcal{I}}$ with $p_C \in \rho(e)$. IH yields $e \in C^{\mathcal{J}}$ and we are done.

  Now assume that $r \notin \Sigma$ and consider the model $\mathcal{J}_{\exists r.C_1,d}$. Since $d \in (\exists r.C_1)^{\mathcal{J}_{\exists r.C_1,d}}$, there is an $e \in C_1^{\mathcal{J}_{\exists r.C_1,d}}$ with $(d,e) \in r^{\mathcal{J}_{\exists r.C_1,d}}$. By definition of $\mathcal{J}$ and Claim 3, we have $(d,e) \in r^{\mathcal{J}}$ and $e \in C_1^{\mathcal{J}}$ and thus we are done.

  This finishes the proof of Claim 4.

Wa can now show that $\mathcal{J}$ is a model of $\mathcal{T}$. Let $C \sqsubseteq D \in \mathcal{T}$. By Claim 3, $d \in C^{\mathcal{J}}$ implies $d \in D^{\mathcal{J}}$ for each $d \in \Delta^{\mathcal{J}} \setminus \Delta^{\mathcal{I}}$. Thus assume $d \in C^{\mathcal{J}}$ with $d \in \Delta^{\mathcal{I}}$. If C does not take the form $\exists r.D$ with $r \notin \Sigma$, then Point 1 of Claim 3 yields $q_D \in \rho(d)$ and thus the transition $q_C \rightarrow p_D$ and Point 2 of Claim 3 yield $d \in C^{\mathcal{I}}$. If $C = \exists r.E$ with $r \notin \Sigma$, then by construction of $\mathcal{J}$ we find a $C' = \exists r.E'$ such that $p_{C'} \in \rho(d)$ and the unique r-successor e of the root of $\mathcal{J}_{C',d}$ is a witness for C at d, that is, $(d,e) \in r^{\mathcal{J}}$ and $e \in C^{\mathcal{J}}$. By Claim 3, $e \in C^{\mathcal{I}}$. Since e is the root of a least tree model for E' and $\mathcal{T}$, we have $\mathcal{T} \models E' \sqsubseteq E$, and thus $\mathcal{T} \models C' \sqsubseteq C$. Due to the transitions $p_{C'} \rightarrow p_C$ and $p_C \rightarrow p_D$, we thus have $p_D \in \rho(d)$ which yields $d \in D^{\mathcal{J}}$ by Point 2 of Claim 3.

To finish the proof of Theorem 24, it thus suffices to show the following.

Claim 5. For all $d \in \Delta^{\mathcal{I}}$, we have $(\mathcal{I}, d) \approx_\Sigma (\mathcal{J}, d)$.

Proof of claim. It suffices to note that the relation

$$S = \{(d,d) \mid d \in \Delta^{\mathcal{I}}\}$$

is a Σ-simulation from $\mathcal{I}$ to $\mathcal{J}$ and also from $\mathcal{J}$ to $\mathcal{I}$. The former is immediate since $A^{\mathcal{I}} \subseteq A^{\mathcal{J}}$ for every $A \in \mathsf{N_R}$ and $r^{\mathcal{I}} \subseteq r^{\mathcal{J}}$ for every $r \in \mathsf{N_R}$. For the latter, let $\mathcal{J}^-$ by the restriction of $\mathcal{J}$ to those elements that are reachable from an element in $\Delta^{\mathcal{I}}$ by following a (possibly empty) sequence of Σ-role edges. It clearly suffices to show that S is a simulation from $\mathcal{J}^-$ to $\mathcal{I}$ which is trivial since $A^{\mathcal{I}} = A^{\mathcal{J}^-}$ for every $A \in \mathsf{N_R}$ and $r^{\mathcal{I}} = r^{\mathcal{J}^-}$ for every $r \in \mathsf{N_R}$.

Theorem 27. Given an $\mathcal{EL}$-TBox $\mathcal{T}$ and signature Σ, it is ExpTime-hard to decide whether there is an $\mathcal{EL}$-TBox that is the uniform $\mathcal{EL}_\Sigma$-interpolant of $\mathcal{T}$.

Proof. We employ the ExpTime-hardness proof for conservative extensions for $\mathcal{EL}$ given in (Lutz and Wolter 2010) which we advise the reader to have at hand. It is proved in (Lutz and Wolter 2010) that for $\mathcal{EL}$-TBoxes $\mathcal{T} \subseteq \mathcal{T}'$ and $B \in \mathsf{sig}(\mathcal{T})$ such that

1. all inclusions in $\mathcal{T}'$ are of the form $C \sqsubseteq A$ with A a concept name,
2. B does not occur on the left-hand-side of inclusions in $\mathcal{T}'$,
3. If $C \sqsubseteq E \in \mathcal{T}' \setminus \mathcal{T}$, then $E = B$ or $E \notin \mathsf{sig}(\mathcal{T})$,
4. $\mathcal{T}'$ is a conservative extension of $\mathcal{T}$ iff there exists a $\mathsf{sig}(\mathcal{T})$-concept C with $\mathcal{T} \not\models C \sqsubseteq B$ and $\mathcal{T}' \models C \sqsubseteq B$;

it is ExpTime-hard to decide whether $\mathcal{T}'$ is a conservative extension of $\mathcal{T}$.

Now assume $\mathcal{T} \subseteq \mathcal{T}'$ with Properties (1.)–(4.) are given. Let A_0, E_0, A_1, E_1 be fresh concept names, r a fresh role name, and set $\Sigma = \mathsf{sig}(\mathcal{T}) \cup \{A_0, E_0, r\}$.

Obtain a TBox $\mathcal{T}_0$ from $\mathcal{T}$ by replacing every occurrence of B by $B \sqcap A_0$ and adding $\{B \sqcap A_0 \sqsubseteq \exists r.E_0, E_0 \sqsubseteq \exists r.E_0\}$ to $\mathcal{T}$. Obtain a TBox $\mathcal{T}_0'$ from $\mathcal{T}'$ by replacing every occurrence of B by $B \sqcap A_1$ and adding $\{B \sqcap A_1 \sqsubseteq \exists r.E_1, E_1 \sqsubseteq \exists r.E_1\}$ to $\mathcal{T}'$.

Claim. $\mathcal{T}'$ is a conservative extension of $\mathcal{T}$ iff $\mathcal{T}_0 \cup \mathcal{T}_0'$ has a $\mathcal{EL}_\Sigma$-uniform interpolant.

The direction ($\Rightarrow$) is straightforward by proving that if $\mathcal{T}'$ is a conservative extension of $\mathcal{T}$ then $\mathcal{T}_0$ is a uniform $\mathcal{EL}_\Sigma$-interpolant of $\mathcal{T}_0 \cup \mathcal{T}_0'$.

Conversely, assume that $\mathcal{T}'$ is not a conservative extension of $\mathcal{T}$. By Point 3, there is a Σ-concept C such that $\mathcal{T}' \models C \sqsubseteq B$ and $\mathcal{T} \not\models C \sqsubseteq B$. Then $\mathcal{T}_0 \cup \mathcal{T}_0' \models C \sqsubseteq B \sqcap A_1$ and, therefore, $\mathcal{T}_0 \cup \mathcal{T}_0' \models C \sqsubseteq \exists r^n.\top$, for all $n \geq 0$. Now, if a uniform $\mathcal{EL}_\Sigma$-interpolant $\mathcal{U}$ exists, then $\mathcal{U} \models C \sqsubseteq B \sqcap A_0$ and, therefore $\mathcal{T}_0 \cup \mathcal{T}_0' \models C \sqsubseteq A_0$. But then $\mathcal{T} \models C \sqsubseteq B$, and we have derived a contradiction. ❏

F Proofs for Section 8

We state the two points of Lemma 31 as separate lemmas.

Lemma 59. *If $\mathcal{I} \in L(\mathcal{A}_\mathcal{T})$, then $\mathcal{I} \models C \sqsubseteq D$ for all $\mathcal{EL}$-CIs $C \sqsubseteq D$ with $\mathcal{T} \models C \sqsubseteq D$.*

Proof. Let $\mathcal{I} \in L(\mathcal{A}_\mathcal{T})$ and let ρ be a run of $\mathcal{A}_\mathcal{T}$ on $\mathcal{I}$. We want to show that for any $\mathcal{EL}$ inclusion $C_0 \sqsubseteq D_0$ with $\mathcal{I} \not\models C_0 \sqsubseteq D_0$, we have $\mathcal{T} \not\models C_0 \sqsubseteq D_0$. W.l.o.g., we assume that $\mathcal{I}$ is a tree model and that $C_0 \sqsubseteq D_0$ is violated at the root d_0 of $\mathcal{T}$. We first establish some technical claims. For every $\Gamma \subseteq \mathsf{dis}(\mathcal{T})$, let $\mathsf{cons}(\Gamma)$ be the set of all sets $S \subseteq \mathsf{sub}(\mathcal{T})$ such that there is a model $\mathcal{I}_S$ of $\mathcal{T}$ and a $d \in \Delta^{\mathcal{I}_S}$ such that $d \in (\bigsqcap \Gamma)^{\mathcal{I}_S}$ and for all $D \in \mathsf{sub}(\mathcal{T})$, we have $d \in D^{\mathcal{I}_S}$ iff $D \in S$. We have

- $\mathcal{T} \models \bigsqcap \Gamma \sqsubseteq \bigsqcup_{S \in \mathsf{cons}(\Gamma)} \bigsqcap S$ and
- $\mathcal{T} \models \bigsqcap S \sqsubseteq \bigsqcap \Gamma$, for all $S \in \mathsf{cons}(\Gamma)$.

The former is immediate by definition of cons. For the latter, let $S \in \mathsf{cons}(\Gamma)$, $C \in \Gamma$, and C' be the top-level DNF of C, i.e., C' is the result of converting C' into DNF where C' is viewed as a propositional formula with concept names and existential restrictions as variables. The existence of the model $\mathcal{I}_S$ implies that for some disjunct $D_1 \sqcap \cdots \sqcap D_k$ of C', we have $D_1, \ldots, D_k \in S$. Therefore, $\mathcal{T} \models \bigsqcap S \sqsubseteq C$.

Claim 1. For all $\Gamma \subseteq \mathsf{dis}(\mathcal{T})$ and $\mathcal{EL}$-concepts $\exists r.D$ with $\mathcal{T} \models \bigsqcap \Gamma \sqsubseteq \exists r.D$, there are $\exists r.D_1, \ldots, \exists r.D_n \in \mathsf{sub}(\mathcal{T})$ such that $\mathcal{T} \models \bigsqcap \Gamma \sqsubseteq \exists r.D_1 \sqcup \cdots \sqcup \exists r.D_n \sqsubseteq \exists r.D$.

Proof of claim. Let $\mathcal{T} \models \bigsqcap \Gamma \sqsubseteq \exists r.D$. It suffices to show that for each $S \in \mathsf{cons}(\Gamma)$, there is an $\exists r.D_S \in S$ such that $\mathcal{T} \models \exists r.D_S \sqsubseteq \exists r.D$. Assume to the contrary that this is not the case, i.e., there is an $S \in \mathsf{cons}(\Gamma)$ such that for each $\exists r.E \in S$, we have $\mathcal{T} \not\models \exists r.E \sqsubseteq \exists r.D$. Then there is, for each $\exists r.E \in S$, a tree model $\mathcal{I}_{\exists r.E}$ of $\mathcal{T}$ that satisfies $\exists r.E$ at the root d_0. Assume w.l.o.g. that the domains of the models $\mathcal{I}_{\exists r.E}$ are pairwise disjoint, except for the shared root d_0. Let $\mathcal{I}$ be the union of all the models $\mathcal{I}_{\exists r.E}$ modified to make true at the root d_0 exactly the concept names in S. Further modify $\mathcal{I}$ by exhaustively applying the following rule: if $A \equiv C_A \in \mathcal{T}$ and $d_0 \in C_A^\mathcal{I}$, then make A true at d_0. To obtain a contradiction to $\mathcal{T} \models C \sqsubseteq \exists r.D$, it remains to show the following:

1. $\mathcal{I}$ is a model of $\mathcal{T}$.
2. $d_0 \in (\bigsqcap \Gamma)^\mathcal{I}$
3. $d_0 \notin (\exists r.D)^\mathcal{I}$.

First for Point 1. Let $A \equiv C_A \in \mathcal{T}$ or $A \sqsubseteq C_A \in \mathcal{T}$ and $d \in A^\mathcal{I}$. If $d \neq d_0$, then we have $d \in C_A^\mathcal{I}$ since each $\mathcal{I}_{\exists r.E}$ is a model of $\mathcal{T}$. It thus remains to deal with the case $d = d_0$. If $d_0 \in A^\mathcal{I}$, then by construction of $\mathcal{I}$, there are two cases. The first one is $A \in S$. Then $C_A \in S$ by definition of $\mathsf{cons}(C)$, which yields $d_0 \in C_A^\mathcal{I}$ by construction of $\mathcal{I}$. The second case is that A was made true at d_0 in the extension step. Let $\mathcal{I}_0, \mathcal{I}_1, \ldots, \mathcal{I}_m = \mathcal{I}$ be the sequence of models produced in this step. It is not hard to show by induction on i that for all $A \equiv C_A \in \mathcal{T}$ with A made true at d_0 in $\mathcal{I}_{i+1}$, we have $d_0 \in C_A^{\mathcal{I}_i}$. It follows that $d_0 \in C_A^\mathcal{I}$ as required. Now let $A \equiv C_A \in \mathcal{T}$ and $d \in C_A^\mathcal{I}$. If $d \neq d_0$, then we have $d \in A^\mathcal{I}$ since each $\mathcal{I}_{\exists r.E}$ is a model of $\mathcal{T}$. If $d = d_0$, then the extension step yields $d_0 \in A^\mathcal{I}$.

For Point 2, the construction of $\mathcal{I}$ yields $d_0 \in (\bigsqcap S)^\mathcal{I}$. Since $\mathcal{T} \models \bigsqcap S \sqsubseteq \bigsqcap \Gamma$, we obtain $d_0 \in (\bigsqcap \Gamma)^\mathcal{I}$. Point 3 is immediate by the semantics of existential restrictions and the construction of $\mathcal{I}$. This finishes the proof of Claim 1.

Claim 2. For all $d \in \Delta^\mathcal{I}$, $C \in \mathsf{dis}(\mathcal{T})$, and $\mathcal{EL}$-concepts D, $p_C \in \rho(d)$ and $\mathcal{T} \models C \sqsubseteq D$ implies $d \in D^\mathcal{I}$.

Proof of claim. The proof is by induction on the role depth of D. For role depth 0, D is a conjunction of concept names $A_1 \sqcap \cdots \sqcap A_n$. For $1 \leq i \leq n$, we have $\mathcal{T} \models C \sqsubseteq A_i$ and thus $A_i \in \mathsf{sub}(\mathcal{T})$ and the transition $p_C \to A_i$ yields $d \in A_i^\mathcal{I}$ as required. For the induction step, assume

$$D = A_1 \sqcap \cdots \sqcap A_k \sqcap \exists r_1.D_1 \sqcap \cdots \sqcap \exists r_n.D_n.$$

We can argue as in the induction start that $d \in A_i^\mathcal{I}$ for $1 \leq i \leq k$. For the existential restrictions, select an $\exists r_i.D_i$ with $1 \leq i \leq n$. By Claim 1, we find existential restrictions $\exists r_i.C_1, \ldots, \exists r_i.C_k \in \mathsf{sub}(\mathcal{T})$ such that

$$\mathcal{T} \models C \sqsubseteq \exists r_i.C_1 \sqcup \cdots \sqcup \exists r_i.C_k \sqsubseteq \exists r_i.D_i.$$

Thus $\mathcal{T} \models C \sqsubseteq \exists r_i.(C_1 \sqcup \cdots \sqcup C_k)$ and the transition $p_C \to \langle r_i \rangle p_{C_1 \sqcup \cdots \sqcup C_k}$ yields a $(d, e) \in r_i^\mathcal{I}$ with $p_{C_1 \sqcup \cdots \sqcup C_k} \in \rho(e)$. We also have $\mathcal{T} \models C_1 \sqcup \cdots \sqcup C_k \sqsubseteq D_i$ and can thus apply IH to obtain $e \in D_i^\mathcal{I}$ and are done. This finishes the proof of Claim 2.

From now on, we will for convenience confuse the set $\rho(d)$ and the conjunction $\bigsqcap\{C \mid q_C \in \rho(d)\}$. Note that the states p_C in $\rho(d)$ are ignored.

Claim 3. For all $\mathcal{EL}$-concepts C and $d \in \Delta^\mathcal{I}$, we have that $\mathcal{T} \models \rho(d) \sqsubseteq C$ implies $d \in C^\mathcal{I}$.

Proof of claim. It clearly suffices to consider concepts C of the form $A \in \mathsf{N_C}$ and $\exists r.D$. For $C = A$ a concept name, the transitions $\bigwedge \rho(d) \to q_A$, $q_A \to p_A$, and $p_A \to A$ yield $d \in A^\mathcal{I}$. It thus remains to consider the case $C = \exists r.D$. Let $\mathcal{T} \models \rho(d) \sqsubseteq \exists r.D$. By Claim 1, we find existential restrictions $\exists r_i.C_1, \ldots, \exists r_i.C_k \in \mathsf{sub}(\mathcal{T})$ such that

$$\mathcal{T} \models \rho(d) \sqsubseteq \exists r.C_1 \sqcup \cdots \sqcup \exists r.C_k \sqsubseteq \exists r.D.$$

By the transition $\bigwedge \rho(d) \to q_{\exists r.C_1 \sqcup \cdots \sqcup \exists r.C_k}$ and $q_{\exists r.C_1 \sqcup \cdots \sqcup \exists r.C_k} \to p_{\exists r.C_1 \sqcup \cdots \sqcup \exists r.C_k}$, we have $p_{\exists r.C_1 \sqcup \cdots \sqcup \exists r.C_k} \in \rho(d)$. Since $\mathcal{T} \models \exists r.C_1 \sqcup \cdots \sqcup \exists r.C_k \sqsubseteq \exists r.(C_1 \sqcup \cdots \sqcup C_k)$, the transition $p_{\exists r.C_1 \sqcup \cdots \sqcup \exists r.C_k} \to \langle r \rangle p_{C_1 \sqcup \cdots \sqcup C_k}$ ensures that there is a $(d, e) \in r^\mathcal{I}$ with $p_{C_1 \sqcup \cdots \sqcup C_k} \in \rho(e)$. By Claim 2 and since $\mathcal{T} \models C_1 \sqcup \cdots \sqcup C_k \sqsubseteq D$, it follows that $e \in D^\mathcal{I}$, thus $d \in (\exists r.D)^\mathcal{I}$ as required. This finishes the proof of Claim 3.

For all $d \in \Delta^\mathcal{I}$, we use $\mathcal{I}|_d$ to denote the restriction of $\mathcal{I}$ to the sub tree interpretation rooted at d. An *extension* of a tree interpretation $\mathcal{J}$ with root e is a tree interpretation $\mathcal{J}'$ with root e' such that $(\mathcal{J}, e) \leq (\mathcal{J}', e')$.

Claim 4. For all $d \in \Delta^\mathcal{I}$ and each concept name A with $d \notin A^\mathcal{I}$, there is an extension $\mathcal{I}_A$ of $\mathcal{I}|_d$ that is a model of $\mathcal{T}$ and satisfies $d \notin A^{\mathcal{I}_A}$.

Proof of claim. A $\mathcal{T}$*-type* is a subset $t \subseteq \mathsf{sub}(\mathcal{T})$ such that the following conditions are satisfied:

1. $C \sqcap D \in t$ iff $C \in t$ and $D \in t$, for all $C \sqcap D \in \mathsf{sub}(\mathcal{T})$;
2. $C \sqcup D \in t$ iff $C \in t$ or $D \in t$, for all $C \sqcup D \in \mathsf{sub}(\mathcal{T})$;
3. for all $A \sqsubseteq C \in \mathcal{T}$, $A \in t$ implies $C \in t$;
4. for all $A \equiv C \in \mathcal{T}$, $A \in t$ iff $C \in t$;
5. if $\exists r.C \in t$ and $\mathcal{T} \models C \sqsubseteq D_1 \sqcup \cdots \sqcup D_n$ with each D_i a *con*junction of concepts from $\mathsf{sub}(\mathcal{T})$, then there is an $i \in \{1, \ldots, n\}$ such that for all conjuncts D of D_i with $\exists r.D \in \mathsf{sub}(\mathcal{T})$, we have $\exists r.D \in t$.

We say that a type is *realized* in an interpretation $\mathcal{H}$ at an $e \in \Delta^\mathcal{H}$ if $t = \{C \in \mathsf{sub}(\mathcal{T}) \mid e \in C^\mathcal{H}\}$. We define a total function $\pi : \Delta^{\mathcal{I}|_d} \to \mathsf{TP}$ such that

($*$) for all $e \in \Delta^{\mathcal{I}|_d}$, $\mathcal{T} \models \pi(e) \sqsubseteq \rho(e)$, i.e., $q_{C_1 \sqcup \cdots \sqcup C_k} \in \rho(e)$ implies that one of $C_1, \ldots, C_k$ is in $\pi(e)$.

Specifically, π is defined as follows:

- Since $d \notin A^\mathcal{I}$, Claim 3 yields $\mathcal{T} \not\models \rho(d) \sqsubseteq A$. Thus, there is a model $\mathcal{H}$ of $\mathcal{T}$ and an $e \in \Delta^\mathcal{H}$ such that $e \in \rho(d)^\mathcal{H} \setminus A^\mathcal{H}$. Choose as $\pi(d)$ the type realized at e in $\mathcal{H}$. It is easy to see that ($*$) is satisfied.
- Let $\pi(e)$ be defined, $(e, e') \in r^\mathcal{I}$, and $\pi(e')$ be undefined. Let $\exists r.C_1, \ldots, \exists r.C_n$ be the existential restrictions in $\mathsf{sub}(\mathcal{T}) \setminus \pi(e)$ that concern the role name r. We want to show that there is a $t' \in \mathsf{TP}$ with $\{C_1, \ldots, C_n\} \cap t' = \emptyset$ and such that $\mathcal{T} \models t' \sqsubseteq \rho(e')$, and then set $\pi'(e) = t'$. Assume to the contrary that there is no such t'. Then $\mathcal{T} \models \rho(e') \sqsubseteq C_1 \sqcup \cdots \sqcup C_n$ and the transition $\bigwedge \rho(e') \to q_{C_1 \sqcup \cdots \sqcup C_n}$ yields $q_{C_1 \sqcup \cdots \sqcup C_n} \in \rho(e')$. The transition $\langle r \rangle q_{C_1 \sqcup \cdots \sqcup C_n} \to q_{\exists r.C_1 \sqcup \cdots \sqcup \exists r.C_n}$ yields $q_{\exists r.C_1 \sqcup \cdots \sqcup \exists r.C_n} \in \rho(e)$, in contradiction to the fact that ($*$) is satisfied and none of the $\exists r.C_1, \ldots, \exists r.C_n$ occurs in $\pi(e)$.

By Condition 5 of types we find, for each $e \in \Delta^{\mathcal{I}|_d}$ and $\exists r.C \in \pi(e)$, a tree model $\mathcal{I}_{e,\exists r.C}$ of $\mathcal{T}$ such that $\exists r.C$ is satisfied at the root e of $\mathcal{I}_{e,\exists r.C}$ and for every $\exists r.D \in \mathsf{sub}(\mathcal{T})$ satisfied at e in $\mathcal{I}_{e,\exists r.C}$, we have $\exists r.D \in \pi(e)$. Let Γ be the set of all models $\mathcal{I}_{e,\exists r.C}$. Assume w.l.o.g. that all models $\mathcal{I}_{e,\exists r.C} \in \Gamma$ share with $\Delta^\mathcal{I}$ only the root and that models from Γ have pairwise disjoint domains, except possibly for the root. Define an interpretation $\mathcal{I}_A$ as follows:

$$\Delta^{\mathcal{I}_A} = \Delta^{\mathcal{I}|_d} \cup \bigcup_{\mathcal{I}_{e,\exists r.C} \in \Gamma} \Delta^{\mathcal{I}_{e,\exists r.C}}$$

$$A^{\mathcal{I}_A} = \{e \in \Delta^{\mathcal{I}|_d} \mid A \in \pi(e)\} \cup \bigcup_{\mathcal{I}_{e,\exists r.C} \in \Gamma} A^{\mathcal{I}_{e,\exists r.C}}$$

$$r^{\mathcal{I}_A} = r^\mathcal{I} \cup \bigcup_{\mathcal{I}_{e,\exists r.C} \in \Gamma} r^{\mathcal{I}_{e,\exists r.C}}$$

It is not hard to prove the following by induction on the structure of C, details are left to the reader:

1. for all $\mathcal{I}_{e,\exists r.C} \in \Gamma$ and $e' \in \Delta^{\mathcal{I}_{e,\exists r.C}} \setminus \{e\}$ and $C \in \mathsf{sub}(\mathcal{T})$, we have $e' \in C^{\mathcal{I}_A}$ iff $e' \in C^{\mathcal{I}_{e,\exists r.C}}$;
2. for all $e \in \Delta^{\mathcal{I}|_d}$ and $C \in \mathsf{sub}(\mathcal{T})$, we have $e \in C^{\mathcal{I}_A}$ iff $C \in \pi(e)$.

Since each $\mathcal{I}_{e,\exists r.C} \in \Gamma$ is a model of $\mathcal{T}$ and by Point 2 above together with Properties 3 and 4 of types, we have that $\mathcal{I}_A$ is a model of $\mathcal{T}$. Since $A \notin \pi(d_0)$ and by Condition 2, we also have $d \notin A^{\mathcal{I}_A}$, as required. This finishes the proof of Claim 4.

Now back to the main proof. We have to convert $\mathcal{I}$ into a model of $\mathcal{T}$ such that d_0 still satisfies C_0, but not D_0. As a first step, we annotate some elements of $\mathcal{I}$ with subconcepts of D_0 that are false at those elements. To start choose, for each $C \in \mathsf{sub}(D_0)$ and each $d \in \Delta^\mathcal{I}$ with $d \notin C^\mathcal{I}$, a concept $w(d, C) \in \mathsf{sub}(D_0)$ that is either a concept name or an existential restriction, occurs as a conjunct in C (which includes the case $w(d, C) = C$), and satisfies $d \notin w(d, C)^\mathcal{I}$. Define a partial function $\mu : \Delta^\mathcal{I} \to \mathsf{sub}(D_0)$, as follows:

1. $\mu(d_0) = w(d_0, D_0)$;
2. if $\mu(d) = \exists r.C$, then for all $(d, e) \in r^\mathcal{I}$, we have $e \notin C^\mathcal{I}$; set $\mu(e) = w(e, C)$ for all those e;

Let $\Delta^\mathcal{I}_{\mathsf{CN}}$ be the set of all $d \in \Delta^\mathcal{I}$ with $\mu(d)$ a concept name and $\Delta^\mathcal{I}_\exists$ the set of all $d \in \Delta^\mathcal{I}$ with $\mu(d)$ an existential restriction. To convert $\mathcal{I}$ into the desired model of $\mathcal{T}$, we attach an additional model $\mathcal{I}_d$ to each $d \in \Delta^\mathcal{I}_\exists$ and replace each subinterpretation $\mathcal{I}|_d$ of $\mathcal{I}$ with $d \in \Delta^\mathcal{I}_{\mathsf{CN}}$ with a suitable extension of $\mathcal{I}|_d$, in the sense of Claim 4. Note that $\mu(e)$ is undefined for all elements in $\mathcal{I}|_d$ except d itself, and thus no conflicts arise for these replacements. More specifically, choose the required models as follows:

- For each $d \in \Delta^\mathcal{I}_{\mathsf{CN}}$ with $\mu(d) = A$, by Claim 4, we find an extension $\mathcal{I}_d$ of $\mathcal{I}|_d$ that is a model of $\mathcal{T}$ and such that $d \notin A^{\mathcal{I}_d}$.
- For each $d \in \Delta^\mathcal{I}_\exists$, $d \notin \mu(d)^\mathcal{I}$ and Claim 3 yield $\mathcal{T} \not\models \rho(d) \sqsubseteq \mu(d)$, and thus we find a tree model $\mathcal{I}_d$ of $\mathcal{T}$ that satisfies $\rho(d)$ at its root d, but not $\mu(d)$.

Assume w.l.o.g. that each model $\mathcal{I}_d$, $d \in \Delta^\mathcal{I}_{\mathsf{CN}}$ shares with $\mathcal{I}$ only the elements in $\Delta^{\mathcal{I}|_d}$, that each model $\mathcal{I}_d$, $d \in \Delta^\mathcal{I}_\exists$ shares with $\mathcal{I}$ only the root d, and that $d \neq d'$ implies that $\Delta^{\mathcal{I}_d} \cap \Delta^{\mathcal{I}_{d'}} = \emptyset$. Define an interpretation $\mathcal{J}'$ as follows:

$$\Delta^{\mathcal{J}'} = \Delta^\mathcal{I} \cup \bigcup_{d \in \Delta^\mathcal{I}_{\mathsf{NC}} \cup \Delta^\mathcal{I}_\exists} \Delta^{\mathcal{I}_d}$$

$$A^{\mathcal{J}'} = A^\mathcal{I} \cup \bigcup_{d \in \Delta^\mathcal{I}_{\mathsf{NC}} \cup \Delta^\mathcal{I}_\exists} A^{\mathcal{I}_d}$$

$$r^{\mathcal{J}'} = r^\mathcal{I} \cup \bigcup_{d \in \Delta^\mathcal{I}_{\mathsf{NC}} \cup \Delta^\mathcal{I}_\exists} r^{\mathcal{I}_d}$$

and let $\mathcal{J}$ be obtained from $\mathcal{J}'$ by exhaustively applying the following rule: if $A \equiv C_A \in \mathcal{T}$ and $d \in C_A^\mathcal{J}$ for some $d \in \Delta^\mathcal{I}_\exists$, then make A true at d. It remains to show the following, as it implies $\mathcal{T} \not\models C_0 \sqsubseteq D_0$:

1. $\mathcal{J}$ is a model of $\mathcal{T}$.
2. $d_0 \in C_0^{\mathcal{J}}$.
3. for all $d \in \Delta^{\mathcal{I}}$ with $\mu(d)$ defined, we have $d \notin \mu(d)^{\mathcal{J}}$.

First for Point 1. Let $A \equiv C_A \in \mathcal{T}$ or $A \sqsubseteq C_A \in \mathcal{T}$ and $d \in A^{\mathcal{J}}$. If $d \in \Delta^{\mathcal{J}} \setminus \Delta^{\mathcal{I}}_{\exists}$, then $d \in A^{\mathcal{J}}$ implies $d \in C_A^{\mathcal{J}}$ since each $\mathcal{I}_d$ is a model of $\mathcal{T}$ and by construction of $\mathcal{J}$. It thus remains to deal with the case that $d \in \Delta^{\mathcal{I}}_{\exists}$. If $d \in A^{\mathcal{J}}$, then by construction of $\mathcal{J}$, there are three cases. The first one is $d \in A^{\mathcal{I}}$. Then the transition $A \to q_A$ yields $q_A \in \rho(d)$, thus $d \in A^{\mathcal{I}_d}$. Since $\mathcal{I}_d$ is a model of $\mathcal{T}$, we have $d \in C_A^{\mathcal{I}_d}$, which gives $d \in C_A^{\mathcal{J}}$ be construction of $\mathcal{J}$. The second case is $d \in A^{\mathcal{I}_d}$, and we can argue as before. The third case is that A was made true at d when $\mathcal{J}'$ was extended to $\mathcal{J}$. Let $\mathcal{J}' = \mathcal{J}_0, \mathcal{J}_1, \ldots, \mathcal{J}_m = \mathcal{J}$ be the sequence of models produced during the extension. It is not hard to show by induction on i that for all $B \equiv C_B \in \mathcal{T}$ with B made true at d in $\mathcal{J}_{i+1}$, we have $d \in C_B^{\mathcal{J}_i}$. It follows that $d \in C_A^{\mathcal{J}}$ as required. Now let $A \equiv C_A \in \mathcal{T}$ and $d \in C_A^{\mathcal{J}}$. Due to the extension step, we then also have $d \in A^{\mathcal{J}}$.

Point 2 is immediate by construction of $\mathcal{J}$ and since $d_0 \in C_0^{\mathcal{I}}$. It thus remains to address Point 3. For each $d \in \Delta^{\mathcal{I}}$ with $\mu(d)$ defined, let $\mathsf{dist}(d)$ be the length of the path from d_0 to d. Moreover, let $m = \max_d \mathsf{dist}(d)$. We show by induction on $m - \mathsf{dist}(d)$ that for all $d \in \Delta^{\mathcal{I}}$ with $\mu(d)$ defined, we have $d \notin \mu(d)^{\mathcal{J}}$. In the induction start, $\mu(d)$ is a concept name A. Then the choice of $\mathcal{I}_d$ and construction of $\mathcal{J}'$ yields $d \notin A^{\mathcal{J}'}$. By construction of $\mathcal{J}$ and since $d \notin \Delta^{\mathcal{I}}_{\exists}$, this yields $d \notin A^{\mathcal{J}}$. In the induction step, $\mu(d)$ is an existential restriction $\exists r.C$. Let $(d,e) \in r^{\mathcal{J}'}$ with $e \in C^{\mathcal{J}'}$. If $e \in \Delta^{\mathcal{I}}$, then $\mu(e)$ is $w(e,C)$ and thus we obtain $e \notin C^{\mathcal{J}'}$ by IH. The only other choice is $e \in \Delta^{\mathcal{I}_d}$, and thus also $(d,e) \in r^{\mathcal{I}_d}$. However, $\mathcal{I}_d$ does not satisfy $\mu(d) = \exists r.C$ at the root, thus $e \notin C^{\mathcal{I}_d}$, which yields $e \notin C^{\mathcal{J}'}$. ❑

To complete the proof of Theorem 33 given in the main paper, it remains to show the following.

Lemma 60. *$\mathcal{T}$ is satisfiable iff there is no $\mathcal{EL}$-approximant of $\mathcal{T}'$.*

Proof.
"⇒" Assume $\mathcal{T}$ is satisfiable. We have

$$\mathcal{T} \models E_i \sqsubseteq \exists s^n.\top,$$

for $i = 1, 2$ and all $n > 0$. It is readily checked that no $\mathcal{EL}$-approximant can axiomatize those inclusions for all $n > 0$. For the proof it is important to observe that since $\mathcal{T}$ is satisfiable there are models of $\mathcal{T}'$ satisfying E_i such that M is not satisfied; one can obtain such interpretations from models of $\mathcal{T}$. It follows that in such interpretations F is not satisfied either. Hence, F cannot be used to axiomatize the inclusions above.

"⇐" This direction is more involved. We show the contrapositive, i.e., if $\mathcal{T}$ is not satisfiable then there is an $\mathcal{EL}$-approximant of $\mathcal{T}'$. Assume $\mathcal{T}$ is not satisfiable. First we show:

Claim 1. $\mathcal{T}' \models D \sqsubseteq M$.

We proceed towards contradiction. Suppose that the claim does not hold, i.e., $D \sqcap \neg M$ is satisfiable w.r.t. $\mathcal{T}'$. We show that $\mathcal{T}$ is satisfiable, which is a contradiction to our assumption that $\mathcal{T}$ is not satisfiable. Since $D \sqcap \neg M$ is satisfiable w.r.t. $\mathcal{T}'$, there is some model $\mathcal{I}$ of $\mathcal{T}'$ such that $(D \sqcap \neg M)^{\mathcal{I}} \neq \emptyset$. We set

$$\Delta^{\mathcal{I}'} = (D \sqcap \neg M)^{\mathcal{I}}$$

and define $\mathcal{I}'$ as the restriction of $\mathcal{I}$ to $\Delta^{\mathcal{I}'}$. We show that $\mathcal{I}'$ is a model of $\mathcal{T}$. To this aim, we distinguish all axioms that can appear in $\mathcal{T}$.

- $A_i \equiv \top \in \mathcal{T}$. Then $A_i \equiv D \sqcup M \in \mathcal{T}'$ and thus by $\mathcal{I} \models \mathcal{T}'$, we have $\mathcal{I} \models A_i \equiv D \sqcup M$. Now $\mathcal{I}' \models A_i \sqsubseteq \top$, since $A_i \sqsubseteq \top$ is a tautology. It remains to show $\mathcal{I}' \models \top \sqsubseteq A_i$, i.e., $\Delta^{\mathcal{I}'} \subseteq A_i^{\mathcal{I}'}$. Suppose $d \in \Delta^{\mathcal{I}'}$. Then by the construction of $\mathcal{I}'$, $d \in (D \sqcap \neg M)^{\mathcal{I}}$. Since $(D \sqcap \neg M)^{\mathcal{I}} \subseteq (D \sqcup M)^{\mathcal{I}}$, $d \in (D \sqcup M)^{\mathcal{I}}$. Then by $\mathcal{I} \models A_i \equiv D \sqcup M$, we have $d \in A_i^{\mathcal{I}}$. Since $A_i^{\mathcal{I}'}$ is defined as the restriction of $A_i^{\mathcal{I}}$ to $\Delta^{\mathcal{I}'}$ and $d \in \Delta^{\mathcal{I}'}$, it follows that $d \in A_i^{\mathcal{I}'}$. Hence $\mathcal{I}' \models \top \sqsubseteq A_i$.
- $A_i \equiv P \in \mathcal{T}$. Then $A_i \equiv (P \sqcap D) \sqcup M \in \mathcal{T}'$ and thus by $\mathcal{I} \models \mathcal{T}'$, we have $\mathcal{I} \models A_i \equiv (P \sqcap D) \sqcup M$.

  Suppose $d \in A_i^{\mathcal{I}'}$. By the construction of $\mathcal{I}'$, this implies $d \in A_i^{\mathcal{I}}$. Then by $\mathcal{I} \models A_i \equiv (P \sqcap D) \sqcup M$, $d \in ((P \sqcap D) \sqcup M)^{\mathcal{I}}$. Since $d \in \Delta^{\mathcal{I}'}$, we have that $d \notin M^{\mathcal{I}}$. Therefore $d \in (P \sqcap D)^{\mathcal{I}}$. Finally by $d \in P^{\mathcal{I}}$ and $d \in \Delta^{\mathcal{I}'}$, we have that $d \in P^{\mathcal{I}'}$.

  Conversely, suppose $d \in P^{\mathcal{I}'}$. By the construction of $\mathcal{I}'$, this implies $d \in P^{\mathcal{I}}$; and by $d \in \Delta^{\mathcal{I}'}$, we have that $d \in D^{\mathcal{I}}$. Hence $d \in (P \sqcap D)^{\mathcal{I}}$. Since $(P \sqcap D)^{\mathcal{I}} \subseteq (P \sqcap D) \sqcup M)^{\mathcal{I}}$, $d \in (P \sqcap D) \sqcup M)^{\mathcal{I}}$. Then by $\mathcal{I} \models A_i \equiv (P \sqcap D) \sqcup M$, $d \in A_i^{\mathcal{I}}$. Finally by $d \in \Delta^{\mathcal{I}'}$, this implies $d \in A_i^{\mathcal{I}'}$.
- $A_i \equiv \neg A_j \in \mathcal{T}$. Then $A_i \equiv (\overline{A}_j \sqcap D) \sqcup M \in \mathcal{T}'$ and thus by $\mathcal{I} \models \mathcal{T}'$, we have $\mathcal{I} \models A_i \equiv (\overline{A}_j \sqcap D) \sqcup M$.

  Suppose $d \in A_i^{\mathcal{I}'}$. By the construction of $\mathcal{I}'$, this implies $d \in A_i^{\mathcal{I}}$. Then by $\mathcal{I} \models A_i \equiv (\overline{A}_j \sqcap D) \sqcup M$ and $d \notin M^{\mathcal{I}}$, $d \in (\overline{A}_j \sqcap D)^{\mathcal{I}}$. Since $d \in \overline{A}_j^{\mathcal{I}}$ and $d \notin M^{\mathcal{I}}$, we have $d \notin A_j^{\mathcal{I}}$. Finally by $d \notin A_j^{\mathcal{I}}$ and $d \in \Delta^{\mathcal{I}'}$, $d \notin A_j^{\mathcal{I}'}$.

  Conversely, let $d \in \Delta^{\mathcal{I}'}$ and $d \notin A_j^{\mathcal{I}'}$. These imply $d \in D^{\mathcal{I}}$ and $d \notin A_j^{\mathcal{I}}$. From these, we obtain that $d \in \overline{A}_j^{\mathcal{I}}$. That is we have $d \in (\overline{A}_j \sqcap D)^{\mathcal{I}}$. Since $(\overline{A}_j \sqcap D)^{\mathcal{I}} \subseteq ((\overline{A}_j \sqcap D) \sqcup M)^{\mathcal{I}}$ and $\mathcal{I} \models A_i \equiv (\overline{A}_j \sqcap D) \sqcup M$, $d \in A_i^{\mathcal{I}}$. Finally by $d \in \Delta^{\mathcal{I}'}$, this implies $d \in A_i^{\mathcal{I}'}$.
- $A_i \equiv B_1 \sqcap B_2 \in \mathcal{T}$. Then $A_i \equiv (B_1 \sqcap B_2 \sqcap D) \sqcup M \in \mathcal{T}'$ and thus by $\mathcal{I} \models \mathcal{T}'$, we have $\mathcal{I} \models A_i \equiv (B_1 \sqcap B_2 \sqcap D) \sqcup M$.
  Suppose $d \in A_i^{\mathcal{I}'}$. By the construction of $\mathcal{I}'$, this implies $d \in A_i^{\mathcal{I}}$. Then by $\mathcal{I} \models A_i \equiv (B_1 \sqcap B_2 \sqcap D) \sqcup M$ and $d \notin M^{\mathcal{I}}$, we obtain $d \in (B_1 \sqcap B_2)^{\mathcal{I}}$. Finally by $d \in \Delta^{\mathcal{I}'}$, this implies $d \in (B_1 \sqcap B_2)^{\mathcal{I}'}$.

Conversely, let $d \in (B_1 \sqcap B_2)^{\mathcal{I}'}$. By the construction of $\mathcal{I}'$, this implies $d \in (B_1 \sqcap B_2)^{\mathcal{I}}$. Then by $d \in D^{\mathcal{I}}$ and $\mathcal{I} \models A_i \equiv (B_1 \sqcap B_2 \sqcap D) \sqcup M$, we obtain $d \in A_i^{\mathcal{I}}$. Finally by $d \in \Delta^{\mathcal{I}'}$, this implies $d \in A_i^{\mathcal{I}'}$.

- Let $A_i \equiv \exists r.A_j \in \mathcal{T}$. Then $A_i \equiv (D \sqcap \exists r.(A_j \sqcap D)) \sqcup M \in \mathcal{T}'$ and thus by $\mathcal{I} \models \mathcal{T}'$, we have $\mathcal{I} \models A_i \equiv (D \sqcap \exists r.(A_j \sqcap D)) \sqcup M$.

  Suppose $d \in A_i^{\mathcal{I}'}$. By the construction of $\mathcal{I}'$, this implies $d \in A_i^{\mathcal{I}}$. Then by $d \notin M^{\mathcal{I}}$ and $\mathcal{I} \models A_i \equiv (D \sqcap \exists r.(A_j \sqcap D)) \sqcup M$, we obtain $d \in (D \sqcap \exists r.(A_j \sqcap D))^{\mathcal{I}}$. This means there is some $e \in \Delta^{\mathcal{I}}$ such that $(d, e) \in r^{\mathcal{I}}$ and $e \in (A_j \sqcap D)^{\mathcal{I}}$. By $d \notin M^{\mathcal{I}}$ and $(d, e) \in r^{\mathcal{I}}$, we have $e \notin M^{\mathcal{I}}$, which implies by $e \in D^{\mathcal{I}}$ that $e \in \Delta^{\mathcal{I}'}$. Then by $d, e \in \Delta^{\mathcal{I}'}$, $(d, e) \in r^{\mathcal{I}}$, and $e \in A_j^{\mathcal{I}}$, we obtain $(d, e) \in r^{\mathcal{I}'}$ and $e \in A_j^{\mathcal{I}'}$. Hence $d \in (\exists r.A_j)^{\mathcal{I}'}$.

  Conversely, let $d \in (\exists r.A_j)^{\mathcal{I}'}$. Then there is some $e \in \Delta^{\mathcal{I}'}$ such that $(d, e) \in r^{\mathcal{I}'}$ and $e \in A_j^{\mathcal{I}'}$. By the construction of $\mathcal{I}'$, these imply that $(d, e) \in r^{\mathcal{I}}$ and $e \in A_j^{\mathcal{I}}$. Since $d, e \in \Delta^{\mathcal{I}'}$, we have $d, e \in D^{\mathcal{I}}$ and thus by $(d, e) \in r^{\mathcal{I}}$, $e \in A_j^{\mathcal{I}}$, and $\mathcal{I} \models A_i \equiv (D \sqcap \exists r.(A_j \sqcap D)) \sqcup M$, we obtain $d \in A_i^{\mathcal{I}}$. Finally by $d \in \Delta^{\mathcal{I}'}$, this implies $d \in A_i^{\mathcal{I}'}$.

Hence we conclude that $\mathcal{I}' \models \mathcal{T}$. But this contradicts with our assumption that $\mathcal{T}$ is not satisfiable. This means that Claim 1 holds.

Now let $\mathcal{T}^\star$ be the TBox consisting of the following $\mathcal{EL}$-CIs:

$$
\begin{aligned}
F &\sqsubseteq M \\
M \sqcap E_i &\sqsubseteq F, \text{ for } i \in \{1,2\} \\
E_i &\sqsubseteq \exists s.F, \text{ for } i \in \{1,2\} \\
F &\sqsubseteq \exists s.F \\
M &\sqsubseteq \bigsqcap_{1 \le i \le n} A_i \\
A_i &\sqsubseteq M, \text{ for } 1 \le i \le n \\
\exists r.M &\sqsubseteq M \\
D &\sqsubseteq M
\end{aligned}
$$

Claim 2. $\mathcal{T}' \models \mathcal{T}^\star$.

We show that for every CI $C \sqsubseteq D \in \mathcal{T}^\star$, we have $\mathcal{T}' \models C \sqsubseteq D$.

- $F \sqsubseteq M$: This is an immediate consequence of $F \equiv (E_1 \sqcup E_2) \sqcap M \in \mathcal{T}'$.
- $M \sqcap E_i \sqsubseteq F$: Again an immediate consequence of $F \equiv (E_1 \sqcup E_2) \sqcap M \in \mathcal{T}'$.
- $E_i \sqsubseteq \exists s.F$: By definition $E_i \sqsubseteq (D \sqcap \exists s.(D \sqcap (E_1 \sqcup E_2))) \in \mathcal{T}'$. This implies

$$\mathcal{T}' \models E_i \sqsubseteq \exists s.(D \sqcap (E_1 \sqcup E_2)) \qquad (1)$$

  Now by Claim 1, $\mathcal{T}' \models D \sqsubseteq M$, and thus we have $\mathcal{T}' \models \exists s.(D \sqcap (E_1 \sqcup E_2)) \sqsubseteq \exists s.(M \sqcap (E_1 \sqcup E_2))$. This and (1) then imply that $\mathcal{T}' \models E_i \sqsubseteq \exists s.(M \sqcap (E_1 \sqcup E_2))$. Finally, we replace $M \sqcap (E_1 \sqcup E_2)$ by F to obtain $\mathcal{T}' \models E_i \sqsubseteq \exists s.F$, since $F \equiv (E_1 \sqcup E_2) \sqcap M \in \mathcal{T}'$.
- $F \sqsubseteq \exists s.F$: By $F \equiv (E_1 \sqcup E_2) \sqcap M \in \mathcal{T}'$, $\mathcal{T}' \models F \sqsubseteq E_1 \sqcup E_2$. Then by $\mathcal{T} \models E_i \sqsubseteq \exists s.F$, we obtain $\mathcal{T}' \models F \sqsubseteq \exists s.F$.
- $M \sqsubseteq \bigsqcap_{1 \le i \le n} A_i$: Let $i \in \{1, \ldots, n\}$. By the definition of $\mathcal{T}'$, there is some CI $A_i \equiv C \sqcup M$ in $\mathcal{T}'$, where C actually depends on the concept C_i such that $A_i \equiv C_i \in \mathcal{T}$. Then by $A_i \equiv C \sqcup M \in \mathcal{T}'$, we immediately obtain that $M \sqsubseteq A_i$, which is what we wanted to show.
- $A_i \sqsubseteq M$. Then by the definition of $\mathcal{T}'$, there is some CI $A_i \equiv (C \sqcap D) \sqcup M$ in $\mathcal{T}'$. Now let $\mathcal{I}$ be a model of $\mathcal{T}'$ and let $s \in \Delta^{\mathcal{I}}$ with $s \in A_i^{\mathcal{I}}$. We will show that $s \in M^{\mathcal{I}}$. By $\mathcal{I} \models A_i \equiv (C \sqcap D) \sqcup M$, either $s \in M^{\mathcal{I}}$ or $s \in D^{\mathcal{I}}$. If former is the case, then we are done; otherwise $s \in D^{\mathcal{I}}$. Then by Claim 1, $s \in M^{\mathcal{I}}$, which is what we wanted to show.
- $\exists r.M \sqsubseteq M$. This is an immediate consequence of $M \equiv \exists r.M \sqcup \bigsqcup_{1 \le i \le n} (\overline{A_i} \sqcap A_i) \in \mathcal{T}'$.
- $D \sqsubseteq M$. We have already shown this in Claim 1.

Hence it follows that $\mathcal{T}' \models \mathcal{T}^\star$. This marks the end of Claim 2.

Now by Claim 2, we have $\mathcal{T}^\star \subseteq^{\mathcal{EL}} \mathcal{T}'$. We claim that $\mathcal{T}^\star$ is an $\mathcal{EL}$-approximant of $\mathcal{T}'$. To this aim, it remains to show that $\mathcal{T}' \subseteq^{\mathcal{EL}} \mathcal{T}^\star$. We show the contrapositive. Suppose the $\mathcal{EL}$-CI does not follow from $\mathcal{T}^\star$, i.e., $\mathcal{T}^\star \not\models C \sqsubseteq D$. We need to show that $\mathcal{T}' \not\models C \sqsubseteq D$. We proceed towards contradiction so suppose that $\mathcal{T}' \models C \sqsubseteq D$. In the rest of the proof, it will be convenient to assume that $\mathcal{T}$ contains at least two axioms, meaning that we have the symbols A_1 and A_2 at our disposal. If $\mathcal{T}$ does not contain at least two axioms, let $\mathcal{T}_1$ be the $\mathcal{EL}^\neg$-TBox that is a result of adding to $\mathcal{T}$ the definitions $A_i \equiv \top$, for $i \in \{1, 2\}$, where A_i does not appear in $\mathcal{T}$. It is not hard to see that $\mathcal{T}$ is satisfiable iff $\mathcal{T}_1$ is satisfiable. Hence our assumption is w.l.o.g.

Since $\mathcal{T}^\star \not\models C \sqsubseteq D$, there is some model $\mathcal{I}_0$ of $\mathcal{T}^\star$ with $\mathcal{I}_0 \not\models C \sqsubseteq D$. W.l.o.g. we assume that $\mathcal{I}_0$ is a tree interpretation with $\rho_0 \in C^{\mathcal{I}_0} \setminus D^{\mathcal{I}_0}$, where ρ_0 is the root of $\mathcal{I}_0$, and $X^{\mathcal{I}_0} = \emptyset$ for all $X \notin \mathsf{sig}(\mathcal{T}')$. This is because we can always unravel a model $\mathcal{I}$ of $\mathcal{T}^\star$ for the individual $d \in \Delta^{\mathcal{I}}$ with $d \in C^{\mathcal{I}} \setminus D^{\mathcal{I}}$. Our aim is to construct a sequence of interpretations $\varsigma = \mathcal{I}_0, \mathcal{I}_1, \ldots$, where $\mathcal{I}_{i+1}$ is obtained from $\mathcal{I}_i$ by fixing a 'minimal defect' in $\mathcal{I}_i$. To be more precise, we need the following definitions.

Let $\mathcal{I}$ be an interpretation in the sequence ς. A *defect* in $\mathcal{I}$ is a $d \in \Delta^{\mathcal{I}}$ that satisfies one of the following conditions:

1. $d \in F^{\mathcal{I}}$ and $d \notin (E_1 \sqcup E_2)^{\mathcal{I}}$;
2. $d \in M^{\mathcal{I}}$, $d \notin (\exists r.M)^{\mathcal{I}}$, and $d \notin \overline{A_i}^{\mathcal{I}}$ for all $i \in \{1, \ldots, n\}$.

A *repair* for a defect d is defined, according to the type of d, as follows:

- For defects of type 1, the repair consists of two copies $\mathcal{I}_1$, $\mathcal{I}_2$ of $\mathcal{I}$ which coincide with $\mathcal{I}$ with the exception that

– $E_1^{\mathcal{I}_1} = E_1^{\mathcal{I}} \cup \{d\}$,
– $E_2^{\mathcal{I}_2} = E_2^{\mathcal{I}} \cup \{d\}$.

- For defects of type 2, the repair consists of two copies $\mathcal{I}_1$, $\mathcal{I}_2$ of $\mathcal{I}$ which coincide with $\mathcal{I}$ with the exception that
  – $\overline{A}_1^{\mathcal{I}_1} = \overline{A}_1^{\mathcal{I}} \cup \{d\}$,
  – $\overline{A}_2^{\mathcal{I}_2} = \overline{A}_2^{\mathcal{I}} \cup \{d\}$.

We remind the reader that the symbols A_1 and A_2 we use in repairs of defects of type 2 appear in $\mathcal{T}'$ because of our assumption about $\mathcal{T}$. A *minimal defect* is a defect d such that there is no defect on the path from the root of $\mathcal{I}$ to d.

Claim 3. Let d_0 be a minimal defect of type 1 in $\mathcal{I}$ and $\mathcal{I}_1, \mathcal{I}_2$ its repairs. Then for all $d \in \Delta^{\mathcal{I}}$, $(\mathcal{I}, d) \approx (\mathcal{I}_1 \times \mathcal{I}_2, (d, d))$.

Let $\mathcal{I}, \mathcal{I}_1, \mathcal{I}_2$, and d_0 be as specified in the lemma.
"$\Rightarrow$" Define $S = \{(d, (d, d)) \mid d \in \Delta^{\mathcal{I}}\}$. We claim that for all $d \in \Delta^{\mathcal{I}}$, $S : (\mathcal{I}, d) \leq (\mathcal{I}_1 \times \mathcal{I}_2, (d, d))$. By $\Delta^{\mathcal{I}} = \Delta^{\mathcal{I}_1} = \Delta^{\mathcal{I}_2}$, we have $S \subseteq \Delta^{\mathcal{I}} \times (\Delta^{\mathcal{I}_1} \times \Delta^{\mathcal{I}_2})$. Moreover, for every $d \in \Delta^{\mathcal{I}}$, $(d, (d, d)) \in S$. It remains to show that (base) and (forth) are satisfied. Let $(d, (d_1, d_2)) \in S$.

- (base). Suppose $d \in A^{\mathcal{I}}$. By the definition of S, we have $d_1 = d_2 = d$ and thus, we need to show that $(d, d) \in A^{\mathcal{I}_1 \times \mathcal{I}_2}$. $d \in A^{\mathcal{I}}$ implies by the definition of $\mathcal{I}_1$ and $\mathcal{I}_2$ that $d \in A^{\mathcal{I}_1}$ and $d \in A^{\mathcal{I}_2}$. Then by the definition of $\mathcal{I}_1 \times \mathcal{I}_2$, $(d, d) \in A^{\mathcal{I}_1 \times \mathcal{I}_2}$.
- (forth). Suppose $(d, e) \in r^{\mathcal{I}}$. By the definition of S, we have $d_1 = d_2 = d$. We need to show that there is some $(e_1, e_2) \in \Delta^{\mathcal{I}_1 \times \mathcal{I}_2}$ such that $((d, d), (e_1, e_2)) \in r^{\mathcal{I}_1 \times \mathcal{I}_2}$ and $(e, (e_1, e_2)) \in S$. Since $\{(e, e)\} = \{(e_1, e_2) \mid (e, (e_1, e_2)) \in S\}$, it is enough to show that $((d, d), (e, e)) \in r^{\mathcal{I}_1 \times \mathcal{I}_2}$. $(d, e) \in r^{\mathcal{I}}$ implies by the definition of $\mathcal{I}_1$ and $\mathcal{I}_2$ that $(d, e) \in r^{\mathcal{I}_1} = r^{\mathcal{I}_2}$. Then by the definition of $\mathcal{I}_1 \times \mathcal{I}_2$, $((d, d), (e, e)) \in r^{\mathcal{I}_1 \times \mathcal{I}_2}$.

Hence it follows that for all $d \in \Delta^{\mathcal{I}}$, we have $S : (\mathcal{I}, d) \leq (\mathcal{I}_1 \times \mathcal{I}_2, (d, d))$.

"$\Leftarrow$" Let $S \subseteq (\Delta^{\mathcal{I}_1} \times \Delta^{\mathcal{I}_2}) \times \Delta^{\mathcal{I}}$ consist of all pairs $((d_1, d_2), d)$ such that

- $\mathsf{level}(d) = \mathsf{level}(d_1) = \mathsf{level}(d_2)$;
- if d_1 is on the path to d_0, then $d_2 = d$;
- if d_2 is on the path to d_0, then $d_1 = d$;
- if neither d_1 nor d_2 are on the path to d_0, then $d \in \{d_1, d_2\}$.

We claim that for all $d \in \Delta^{\mathcal{I}}$, $S : (\mathcal{I}_1 \times \mathcal{I}_2, (d, d)) \leq (\mathcal{I}, d)$. Let $d \in \Delta^{\mathcal{I}}$. By the definition of S, we have $((d, d), d) \in S$. It remains to show that (base) and (forth) are satisfied. Let $((d_1, d_2), d) \in S$.

- (base) Suppose $(d_1, d_2) \in A^{\mathcal{I}_1 \times \mathcal{I}_2}$. Then by the definition of a product, $d_1 \in A^{\mathcal{I}_1}$ and $d_2 \in A^{\mathcal{I}_2}$; and by the definition of S, $\mathsf{level}(d) = \mathsf{level}(d_1) = \mathsf{level}(d_2)$. We distinguish cases.
  – d_1 is on the path to d_0. This means $d_2 = d$. We proceed towards contradiction so suppose that $d \notin A^{\mathcal{I}}$. By the definition of $\mathcal{I}_2$ and $d \in A^{\mathcal{I}_2}$, this means that $d = d_0$ and $A = E_2$. Then by $d_1 \in A^{\mathcal{I}_1}$, we obtain $d_1 \in E_2^{\mathcal{I}_1}$. Since $E_2^{\mathcal{I}_1} = E_2^{\mathcal{I}}$, $d_1 \in E_2^{\mathcal{I}}$. This implies that $d_1 \neq d_0$, since d_0 is a type 1 defect in $\mathcal{I}$ and thus, $d_0 \notin (E_1 \sqcup E_2)^{\mathcal{I}}$. Since d_1 is on the path to d_0, $d_1 \neq d_0$ implies that $\mathsf{level}(d_1) < \mathsf{level}(d_0)$. But this contradicts with $\mathsf{level}(d) = \mathsf{level}(d_1)$. Hence $d \in A^{\mathcal{I}}$.
  – d_2 is on the path to d_0. Analogous to the previous case.
  – Neither d_1 nor d_2 are on the path to d_0. This implies that $d_1 \neq d_0$ and $d_2 \neq d_0$. By assumption we have that $d \in \{d_1, d_2\}$. Suppose first that $d = d_1$. Then by $d_1 \in A^{\mathcal{I}_1}$ and $d_1 \neq d_0$, we have $d_1 \in A^{\mathcal{I}}$. Finally by $d = d_1$, $d \in A^{\mathcal{I}}$. The case for $d = d_2$ can be shown analogously.

  Hence we conclude that (base) is satisfied.
- (forth) Suppose $((d_1, d_2), (e_1, e_2)) \in r^{\mathcal{I}_1 \times \mathcal{I}_2}$. We need to show that there is some $e \in \Delta^{\mathcal{I}}$ such that $(d, e) \in r^{\mathcal{I}}$ and $((e_1, e_2), e) \in S$. By $((d_1, d_2), (e_1, e_2)) \in r^{\mathcal{I}_1 \times \mathcal{I}_2}$, we obtain $(d_1, e_1) \in r^{\mathcal{I}_1}$ and $(d_2, e_2) \in r^{\mathcal{I}_2}$. Since $r^{\mathcal{I}} = r^{\mathcal{I}_1} = r^{\mathcal{I}_2}$, we have that $(d_1, e_1) \in r^{\mathcal{I}}$ and $(d_2, e_2) \in r^{\mathcal{I}}$. We distinguish cases.
  – d_1 is on the path to d_0. Then $d_2 = d$. By $(d_2, e_2) \in r^{\mathcal{I}}$, we have $(d, e_2) \in r^{\mathcal{I}}$. If we show that $((e_1, e_2), e_2) \in S$ then we are done. Now by $(d_1, e_1) \in r^{\mathcal{I}}$ and the fact that d_1 is on the path to d_0, we have the following cases.
    * e_1 is also on the path to d_0. Then by the definition of S, it immediately follows that $((e_1, e_2), e_2) \in S$, which is what we wanted to show.
    * e_1 is not on the path to d_0 because $d_1 = d_0$. Then by $\mathsf{level}(d_1) = \mathsf{level}(d_2) = \mathsf{level}(d_0)$ and $(d_2, e_2) \in r^{\mathcal{I}}$, we have that $\mathsf{level}(e_2) = \mathsf{level}(d_0) + 1$. Hence e_2 is also not on the path to d_0. Since neither e_1 nor e_2 are on the path to d_0, we obtain by the definition of S that $((e_1, e_2), e_2) \in S$, which is what we wanted to show.
  – d_2 is on the path to d_0. Analogous to the previous case.
  – Neither d_1 nor d_2 are on the path to d_0. Then $d \in \{d_1, d_2\}$. First suppose that $d = d_1$. This implies by $(d_1, e_1) \in r^{\mathcal{I}}$ that $(d, e_1) \in r^{\mathcal{I}}$. If we show that $((e_1, e_2), e_1) \in S$ then we are done. By the fact that for $i \in \{1, 2\}$, d_i is not on the path to d_0 and $(d_i, e_i) \in r^{\mathcal{I}}$, e_i is also not on the path to d_0. But then $((e_1, e_2), e_1) \in S$, which is what we wanted to show. The case for $d = d_2$ can be shown analogously.

  Hence we conclude that (forth) is satisfied.

This marks the end of the proof of Claim 3. The proof of the next claim is very similar to that of Claim 3, so we omit it here.

Claim 4. Let d_0 be a minimal defect of type 2 in $\mathcal{I}$ and $\mathcal{I}_1, \mathcal{I}_2$ its repairs. Then for all $d \in \Delta^{\mathcal{I}}$, $(\mathcal{I}, d) \approx (\mathcal{I}_1 \times \mathcal{I}_2, (d, d))$.

Now we start from $\mathcal{I}_0$ and do the following. We choose a minimal defect d_0 in $\mathcal{I}_0$, if it has any. Let $\mathcal{I}_a$ and $\mathcal{I}_b$ be the repairs of this defect. By Claim 3 and Claim 4, for all $d \in \Delta_0^{\mathcal{I}}$, we have that $(\mathcal{I}_0, d) \approx (\mathcal{I}_a \times \mathcal{I}_b, (d, d))$. By $\rho_0 \in C^{\mathcal{I}_0} \setminus D^{\mathcal{I}_0}$, this implies $(\rho_0, \rho_0) \in C^{\mathcal{I}_a \times \mathcal{I}_b} \setminus D^{\mathcal{I}_a \times \mathcal{I}_b}$. Hence by Lemma 8, we obtain $\rho_0 \in C^{\mathcal{I}_a}$, $\rho_0 \in C^{\mathcal{I}_b}$, and either

$\rho_0 \notin D^{\mathcal{I}_a}$ or $\rho_0 \notin D^{\mathcal{I}_b}$. Let $c \in \{a, b\}$ such that $\rho_0 \notin D^{\mathcal{I}_c}$. Set $\mathcal{I}_1 = \mathcal{I}_c$. Obviously, $\mathcal{I}_1 \not\models C \sqsubseteq D$ and $\mathcal{I}_1$ lacks the defect d_0. Now we proceed inductively, just as the case from $\mathcal{I}_0$ to $\mathcal{I}_1$, to obtain the sequence $\varsigma = \mathcal{I}_0, \mathcal{I}_1, \ldots$. The interpretation $\mathcal{I}$ in the limit of this construction satisfies the following properties:

- $\Delta^{\mathcal{I}} = \Delta^{\mathcal{I}_0}$;
- $E_1^{\mathcal{I}} \supseteq E_1^{\mathcal{I}_0}$ and $E_2^{\mathcal{I}} \supseteq E_2^{\mathcal{I}_0}$;
- $F^{\mathcal{I}_0} = F^{\mathcal{I}} \subseteq E_1^{\mathcal{I}} \cup E_2^{\mathcal{I}}$;
- $\overline{A}_1^{\mathcal{I}} \supseteq \overline{A}_1^{\mathcal{I}_0}$ and $\overline{A}_2^{\mathcal{I}} \supseteq \overline{A}_2^{\mathcal{I}_0}$;
- for all $d \in \Delta^{\mathcal{I}}$, if $d \in M^{\mathcal{I}}$ then either $d \in (\exists r.M)^{\mathcal{I}}$ or $d \in \overline{A}_i^{\mathcal{I}}$ for some $i \in \{1, \ldots, n\}$;
- for all $P \in (\mathsf{N_C} \cup \mathsf{N_R}) \setminus \{E_1, E_2, \overline{A}_1, \overline{A}_2\}$, $P^{\mathcal{I}_0} = P^{\mathcal{I}}$;
- $D^{\mathcal{I}} = D^{\mathcal{I}_0}$.

All the items above are immediate consequences of the construction of $\mathcal{I}$, except the last one. Therefore we now show that

$$D^{\mathcal{I}} = D^{\mathcal{I}_0} \tag{2}$$

That $D^{\mathcal{I}_0} \subseteq D^{\mathcal{I}}$ is easy to see because the extension of the concept names $A_1, \ldots, A_n$ are the same in $\mathcal{I}_0$ and $\mathcal{I}$, whereas $\overline{A}_1^{\mathcal{I}_0} \subseteq \overline{A}_1^{\mathcal{I}}$ and $\overline{A}_2^{\mathcal{I}_0} \subseteq \overline{A}_2^{\mathcal{I}}$. For $D^{\mathcal{I}} \subseteq D^{\mathcal{I}_0}$, we proceed towards contradiction. Suppose there is some $d \in \Delta^{\mathcal{I}} = \Delta^{\mathcal{I}_0}$ with $d \in D^{\mathcal{I}}$ and $d \notin D^{\mathcal{I}_0}$. The latter means that there is some $i \in \{1, \ldots, n\}$ such that $d \notin A_i^{\mathcal{I}_0}$ and $d \notin \overline{A}_i^{\mathcal{I}_0}$. Then by $\mathcal{I}_0 \models \mathcal{T}^\star$, $M \sqsubseteq \bigsqcap_{1 \le i \le n} A_i \in \mathcal{T}^\star$, and $d \notin A_i^{\mathcal{I}_0}$, we obtain that $d \notin M^{\mathcal{I}_0}$. Hence d is not a type 2 defect in $\mathcal{I}_0$. But then $d \notin A_i^{\mathcal{I}}$ and $d \notin \overline{A}_i^{\mathcal{I}}$, i.e., $d \notin D^{\mathcal{I}}$, which is a contradiction.

Now using these properties of $\mathcal{I}$, we show that $\mathcal{I} \models \mathcal{T}'$. To this aim, we show that $\mathcal{I}$ satisfies every CI in $\mathcal{T}'$.

- $A_i \equiv C$, where $C = (C' \sqcap D) \sqcup M$. By $\mathcal{T}^\star \models A_i \sqsubseteq M$ and $\mathcal{I}_0 \models \mathcal{T}^\star$, we have $A_i^{\mathcal{I}_0} \subseteq M^{\mathcal{I}_0}$. This implies that $A_i^{\mathcal{I}} \subseteq M^{\mathcal{I}}$, since $A_i^{\mathcal{I}} = A_i^{\mathcal{I}_0}$ and $M^{\mathcal{I}} = M^{\mathcal{I}_0}$. Finally by $C = (C' \sqcap D) \sqcup M$ we obtain that $\mathcal{I} \models A_i \sqsubseteq C$. It remains to show that $\mathcal{I} \models C \sqsubseteq A_i$. To this aim, suppose $d \in C^{\mathcal{I}}$. Then either $d \in (C' \sqcap D)^{\mathcal{I}}$ or $d \in M^{\mathcal{I}}$. We distinguish these cases.
  - $d \in M^{\mathcal{I}}$. Then by $M \sqsubseteq \bigsqcap_{1 \le i \le n} A_i \in \mathcal{T}^\star$, $\mathcal{I}_0 \models \mathcal{T}^\star$, $A_i^{\mathcal{I}} = A_i^{\mathcal{I}_0}$, and $M^{\mathcal{I}} = M^{\mathcal{I}_0}$, we have $d \in A_i^{\mathcal{I}}$.
  - $d \in (C' \sqcap D)^{\mathcal{I}}$. By $d \in D^{\mathcal{I}}$ and (2), we have $d \in D^{\mathcal{I}_0}$. Then by $D \sqsubseteq M \in \mathcal{T}^\star$ and $\mathcal{I}_0 \models \mathcal{T}^\star$, $d \in M^{\mathcal{I}_0}$. But then $d \in A_i^{\mathcal{I}_0}$, by $\mathcal{I}_0 \models \mathcal{T}^\star$. Finally by $A_i^{\mathcal{I}_0} = A_i^{\mathcal{I}}$, we obtain that $d \in A_i^{\mathcal{I}}$.

  Hence in both cases, we have $d \in A_i^{\mathcal{I}}$, which is what we wanted to show. Therefore, we conclude that $\mathcal{I} \models C \sqsubseteq A_i$.
- $M \equiv \exists r.M \sqcup \bigsqcup_{1 \le i \le n} (\overline{A_i} \sqcap A_i)$. First we show the direction from right to left. Suppose $d \in \left(\exists r.M \sqcup \bigsqcup_{1 \le i \le n} (\overline{A_i} \sqcap A_i)\right)^{\mathcal{I}}$. Then (i) $d \in (\exists r.M)^{\mathcal{I}}$ or (ii) $d \in \left(\bigsqcup_{1 \le i \le n} (\overline{A_i} \sqcap A_i)\right)^{\mathcal{I}}$. If (i) then by $r^{\mathcal{I}} = r^{\mathcal{I}_0}$ and $M^{\mathcal{I}} = M^{\mathcal{I}_0}$, we have $(\exists r.M)^{\mathcal{I}} = (\exists r.M)^{\mathcal{I}_0}$, i.e., $d \in (\exists r.M)^{\mathcal{I}_0}$. This implies $d \in M^{\mathcal{I}_0}$, by $\mathcal{I} \models \mathcal{T}^\star$ and $\exists r.M \sqsubseteq M \in \mathcal{T}^\star$. Then by $M^{\mathcal{I}} = M^{\mathcal{I}_0}$ again, we obtain that $d \in M^{\mathcal{I}}$, which is what we wanted to show. Now suppose that (ii) holds. Then there is some $i \in \{1, \ldots, n\}$ such that $d \in (\overline{A}_i \sqcap A_i)^{\mathcal{I}}$. By $d \in A_i^{\mathcal{I}}$ and $A_i^{\mathcal{I}} = A_i^{\mathcal{I}_0}$, we have $d \in A_i^{\mathcal{I}}$. Then by $\mathcal{I}_0 \models A_i \sqsubseteq M$, we obtain $d \in M^{\mathcal{I}_0}$. Finally by $M^{\mathcal{I}} = M^{\mathcal{I}_0}$, we get the desired result that $d \in M^{\mathcal{I}}$, which is what we wanted to show.

  Conversely, we make use of the fact that $\mathcal{I}$ is defect-free. Suppose $d \in M^{\mathcal{I}}$. This means $d \in M^{\mathcal{I}_0}$, by $M^{\mathcal{I}} = M^{\mathcal{I}_0}$. If d is a not type 2 defect in $\mathcal{I}_0$ then the desired result follows. Otherwise, suppose that d is a type 2 defect in $\mathcal{I}_0$. By $d \in M^{\mathcal{I}_0}$, we have $d \in A_i^{\mathcal{I}_0}$ for all $i \in \{1, \ldots, n\}$. This means $d \in A_i^{\mathcal{I}}$ for all $i \in \{1, \ldots, n\}$, by the definition of $\mathcal{I}$. So it remains to show that for some $i \in \{1, \ldots, n\}$, we have $d \in \overline{A}_i^{\mathcal{I}}$. Since d is a type 2 defect in $\mathcal{I}_0$, we have by the definition of $\mathcal{I}$ that either $d \in \overline{A}_1^{\mathcal{I}}$ or $d \in \overline{A}_2^{\mathcal{I}}$. But this is what we wanted to show.
- $F \equiv (E_1 \sqcup E_2) \sqcap M$. Suppose $d \in F^{\mathcal{I}}$. Then by the definition of $\mathcal{I}$, we have $d \in (E_1 \sqcup E_2)^{\mathcal{I}}$; and by $F \sqsubseteq M \in \mathcal{T}^\star$, $\mathcal{I}_0 \models \mathcal{T}^\star$, and the fact that the extension of F and M are identical in $\mathcal{I}_0$ and $\mathcal{I}$, we have $d \in M^{\mathcal{I}}$. Hence $\mathcal{I} \models F \sqsubseteq (E_1 \sqcup E_2) \sqcap M$. Conversely, suppose that $d \in M^{\mathcal{I}}$ and $d \in (E_1 \sqcup E_2)^{\mathcal{I}}$. By the former and the definition of $\mathcal{I}$, we have $d \in M^{\mathcal{I}_0}$; and by the latter, we have that $d \in E_1^{\mathcal{I}}$ or $d \in E_2^{\mathcal{I}}$. Suppose first $d \in E_1^{\mathcal{I}}$. We distinguish cases:
  - $d \in E_1^{\mathcal{I}_0}$: Then by $M \sqcap E_1 \sqsubseteq F \in \mathcal{T}^\star$ and $\mathcal{I}_0 \models \mathcal{T}^\star$, we obtain that $d \in F^{\mathcal{I}_0}$. This implies $d \in F^{\mathcal{I}}$ by $F^{\mathcal{I}} = F^{\mathcal{I}_0}$.
  - $d \notin E_1^{\mathcal{I}_0}$: This means $d \in F^{\mathcal{I}_0}$ by the definition of a defect. Then by $F^{\mathcal{I}} = F^{\mathcal{I}_0}$, we obtain $d \in F^{\mathcal{I}}$, which is what we wanted to show.

  The case for $d \in E_2^{\mathcal{I}}$ is analogous. Hence we conclude that $\mathcal{I} \models (E_1 \sqcup E_2) \sqcap M \sqsubseteq F$.

Thus, $\mathcal{I} \models \mathcal{T}'$. Then by $\mathcal{T}' \models C \sqsubseteq D$, we obtain $\mathcal{I} \models C \sqsubseteq D$. But this is a contradiction since with our inductive construction of $\mathcal{I}$, we guaranteed that $\mathcal{I} \not\models C \sqsubseteq D$. Hence $\mathcal{T}' \subseteq^{\mathcal{EL}} \mathcal{T}^\star$, which is what we wanted to show.

❑