Ontology Languages (COMP321)

Solutions for Exercise 4

1. Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be an interpretation defined by

   - $\Delta^{\mathcal{I}} = \{2, 3, 4\}$;
   - $A^{\mathcal{I}} = \{2\}$;
   - $B^{\mathcal{I}} = \{3, 4\}$;
   - $r^{\mathcal{I}} = \{(2, 2), (2, 3), (4, 2)\}$.

   Determine

   - $(A \sqcup B)^{\mathcal{I}}$;
   - $(A \sqcap B)^{\mathcal{I}}$;
   - $(\top \sqcap \neg(A \sqcup \neg B))^{\mathcal{I}}$;
   - $(\forall r.(A \sqcup B))^{\mathcal{I}}$;
   - $(\forall r.(A \sqcap B))^{\mathcal{I}}$;
   - $(\forall r.A \sqcap \exists r.B)^{\mathcal{I}}$.

   Which of the following statements are true:

   - $\mathcal{I} \models B \sqsubseteq \neg A$?
   - $\mathcal{I} \models \exists r.A \sqcap B \sqsubseteq \forall r.A$?
   - $\mathcal{I} \models \exists r.B \sqsubseteq A$?

2. Apply the $\mathcal{ALC}$-tableau algorithm to the following concepts and determine which are satisfiable and which are not. If a concept is satisfiable, give an interpretation satisfying it.

   - $A \sqcap \neg A$
   - $\exists r.\exists r.(A \sqcap \neg A)$
   - $\forall r.\forall r.(A \sqcap \neg A)$
   - $\exists r.A \sqcap \forall s.\neg A$
   - $\exists r.A \sqcap (\forall r.\neg A \sqcup \exists r.\neg A)$

3. Use the $\mathcal{ALC}$-tableau algorithm to determine whether $\emptyset \models \forall r.A \sqsubseteq \exists r.A$ (in words: determine whether the concept inclusion $\forall r.A \sqsubseteq \exists r.A$ follows from the empty TBox).

Solution for 1:

- $(A \sqcup B)^{\mathcal{I}} = \Delta^{\mathcal{I}}$;
- $(A \sqcap B)^{\mathcal{I}} = \emptyset$;
- $(\top \sqcap \neg(A \sqcup \neg B))^{\mathcal{I}} = \Delta^{\mathcal{I}} \cap \{3,4\} = \{3,4\}$;
- $(\forall r.(A \sqcup B))^{\mathcal{I}} = \Delta^{\mathcal{I}}$;
- $(\forall r.(A \sqcap B))^{\mathcal{I}} = \{3\}$;
- $(\forall r.A \sqcap \exists r.B)^{\mathcal{I}} = \{3,4\} \cap \{2\} = \emptyset$.

Which of the following statements are true?

- $\mathcal{I} \models B \sqsubseteq \neg A$ is true because

$$B^{\mathcal{I}} = \{3,4\} \subseteq \{3,4\} = (\Delta^{\mathcal{I}} \setminus A^{\mathcal{I}}) = (\neg A)^{\mathcal{I}}.$$

- $\mathcal{I} \models B \sqcap \exists r.A \sqsubseteq \forall r.A$ is true: we have $(\exists r.A)^{\mathcal{I}} = \{2,4\}$ and so $(B \sqcap \exists r.A)^{\mathcal{I}} = \{4\}$. We also have $(\forall r.A)^{\mathcal{I}} = \{3,4\}$. Thus

$$(B \sqcap \exists r.A)^{\mathcal{I}} = \{4\} \subseteq \{3,4\} = (\forall r.A)^{\mathcal{I}}$$

- $\mathcal{I} \models \exists r.B \sqsubseteq A$ holds since $(\exists r.B)^{\mathcal{I}} = \{2\} \subseteq A^{\mathcal{I}}$.

Solution for 2.

(a) We check whether $A \sqcap \neg A$ is satisfiable. It is in NNF, so we can directly apply the tableau algorithm to

$$S_0 = \{x : A \sqcap \neg A\}$$

An application of $\rightarrow_{\sqcap}$ gives

$$S_1 = S_0 \cup \{x : A, x : \neg A\}$$

which is a clash. Thus, $A \sqcap \neg A$ is not satisfiable.

(b) We check whether $\exists r.\exists r.(A \sqcap \neg A)$ is satisfiable. It is in NNF, so we can directly apply the tableau algorithm to

$$S_0 = \{x : \exists r.\exists r.(A \sqcap \neg A)\}$$

An application of $\rightarrow_\exists$ gives

$$S_1 = S_0 \cup \{(x, y) : r, y : \exists r.(A \sqcap \neg A)\}$$

An application of $\rightarrow_\exists$ gives

$$S_2 = S_1 \cup \{(y, z) : r, z : (A \sqcap \neg A)\}$$

An application of $\rightarrow_\sqcap$ gives

$$S_3 = S_2 \cup \{z : A, z : \neg A\}$$

which is a clash. Thus, $\exists r.\exists r.(A \sqcap \neg A)$ is not satisfiable.

(c) We check whether $\forall r.\forall r.(A \sqcap \neg A)$ is satisfiable. It is in NNF, so we can directly apply the tableau algorithm to

$$S_0 = \{x : \forall r.\forall r.(A \sqcap \neg A)\}$$

However, no rule is applicable to S_0. S_0 contains no clash. So $\forall r.\forall r.(A \sqcap \neg A)$ is satisfiable and an interpretation $\mathcal{I}$ satisfying $\forall r.\forall r.(A \sqcap \neg A)$ is given by

- $\Delta^\mathcal{I} = \{x\}$;
- $A^\mathcal{I} = \emptyset$;
- $r^\mathcal{I} = \emptyset$.

Then $x \in (\forall r.\forall r.A)^\mathcal{I}$.

(d) We check whether $\exists r.A \sqcap \forall s.\neg A$ is satisfiable. It is in NNF, so we can directly apply the tableau algorithm to

$$S_0 = \{x : \exists r.A \sqcap \forall s.\neg A\}$$

An application of $\rightarrow_\sqcap$ gives

$$S_1 = S_0 \cup \{x : \exists r.A, x : \forall s.\neg A\}.$$

An application of $\rightarrow_\exists$ gives

$$S_2 = S_1 \cup \{(x, y) : r, y : A\}$$

No rule is applicable to S_2. S_2 contains no clash. Thus $\exists r.A \sqcap \forall s.\neg A$ is satisfiable and an interpretation $\mathcal{I}$ satisfying it is given by

- $\Delta^\mathcal{I} = \{x, y\}$;
- $A^\mathcal{I} = \{y\}$;
- $r^\mathcal{I} = \{(x, y)\}$;
- $s^\mathcal{I} = \emptyset$.

(e) We check whether $\exists r.A \sqcap (\forall r.\neg A \sqcup \exists r.\neg A)$ is satisfiable. It is in NNF, so we can directly apply the tableau algorithm to

$$S_0 = \{x : \exists r.A \sqcap (\forall r.\neg A \sqcup \exists r.\neg A)\}$$

An application of $\rightarrow_\sqcap$ gives

$$S_1 = S_0 \cup \{x : \exists r.A, x : (\forall r.\neg A \sqcup \exists r.\neg A)\}$$

An application of $\rightarrow_\exists$ gives

$$S_2 = S_1 \cup \{(x, y) : r, y : A\}$$

Now we have a choice. The first possible application of $\rightarrow_\sqcup$ gives

$$S_3 = S_2 \cup \{x : \forall r.\neg A\}$$

Then an application of $\rightarrow_\forall$ gives

$$S_4 = S_3 \cup \{y : \neg A\}$$

and we have obtained a clash. So this option is unsuccessful. Next we try

$$S_3^* = S_2 \cup \{x : \exists r.\neg A\}$$

An application of $\rightarrow_\exists$ gives

$$S_4^* = S_3^* \cup \{(x, z) : r, z : \neg A\}$$

No rule is applicable to S_4^* and it contains no clash. Thus $\exists r.A \sqcap (\forall r.\neg A \sqcup \exists r.\neg A)$ is satisfiable and an interpretation $\mathcal{I}$ satisfying it is given by

- $\Delta^{\mathcal{I}} = \{x, y, z\}$;
- $A^{\mathcal{I}} = \{y\}$;
- $r^{\mathcal{I}} = \{(x, y), (x, z)\}$.

Solution for 3. We know that $\emptyset \models \forall r.A \sqsubseteq \exists r.A$ if, and only if

$$\forall r.A \sqcap \neg\exists r.A$$

is not satisfiable. Thus, we have check whether $\forall r.A \sqcap \neg\exists r.A$ is satisfiable. Transformation into NNF gives

$$\forall r.A \sqcap \forall r.\neg A$$

We have checked satisfiable of very similar concepts already. Let us take

$$S_0 = \{x : \forall r.A \sqcap \forall r.\neg A\}$$

An application of $\rightarrow_{\sqcap}$ gives

$$S_1 = S_0 \cup \{x : \forall r.A, x : \forall r.\neg A\}$$

No rule is applicable to S_1 and it does not contain a clash. Thus, $\forall r.A \sqcap \forall r.\neg A$ is satisfiable. An interpretation satisfying it is given by

- $\Delta^{\mathcal{I}} = \{x\}$;
- $A^{\mathcal{I}} = \emptyset$;
- $r^{\mathcal{I}} = \emptyset$.

Then $x \in (\forall r.A \sqcap \forall r.\neg A)^{\mathcal{I}}$. It follows that $\emptyset \not\models \forall r.A \sqsubseteq \exists r.A$.