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Simple and Fast Approximate Counting and Leader Election in Populations

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Joint work with
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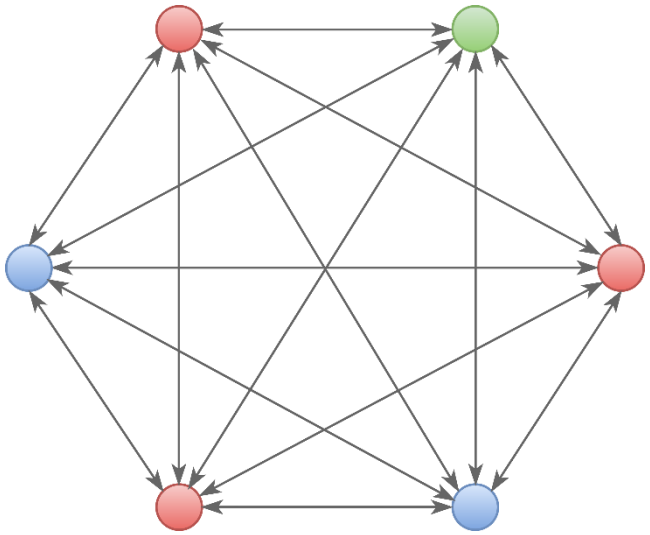
Population Protocols

- Distributed computing model formed by n resource-limited mobile agents
  - Interact in pairs
  - Cannot control their interactions
    - Passive mobility, like particles in a well-mixed solution.
    - A pair of agents interacts in every discrete step under a *uniform random scheduler*.
    - Complete communication graph $G = (V, E)$
- Anonymous (i.e., do not have unique IDs) agents.
- Finite set of states Q.
- Transition function $\delta: Q \times Q \rightarrow Q \times Q$.



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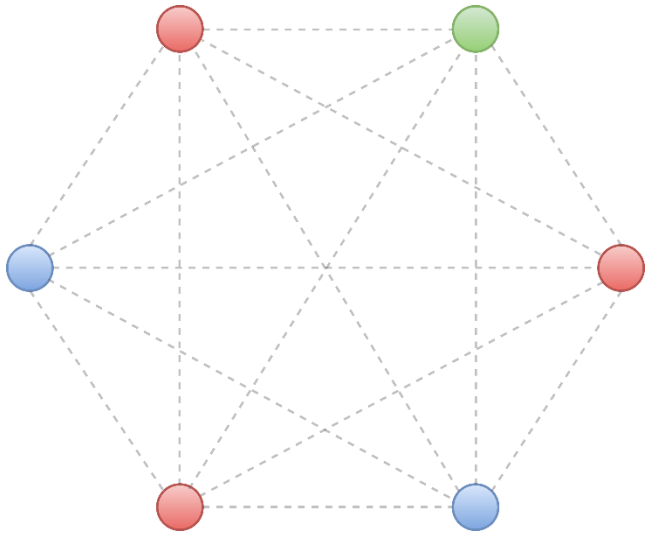

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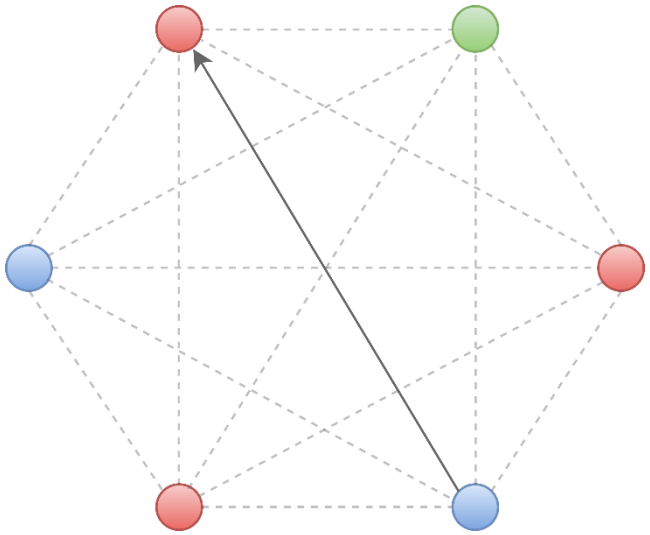

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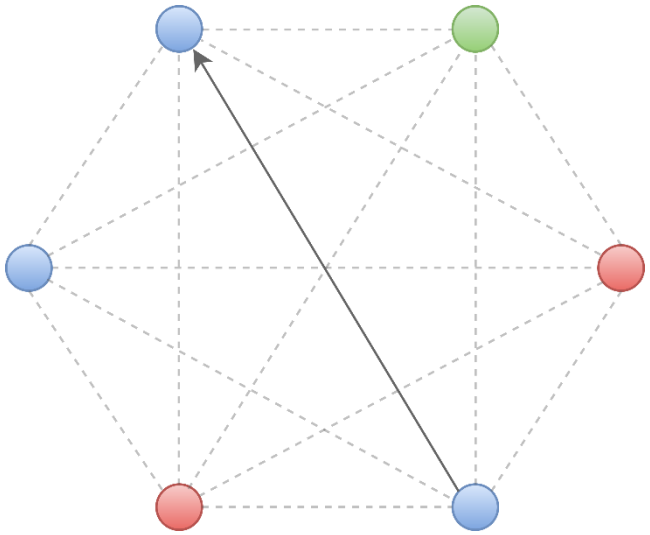

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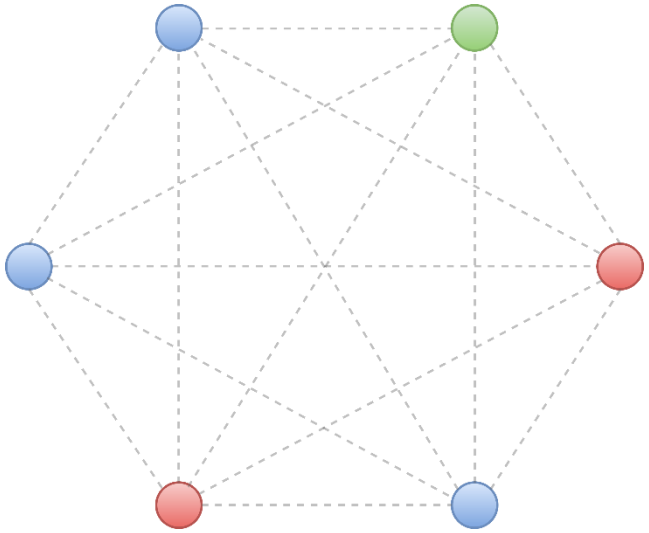

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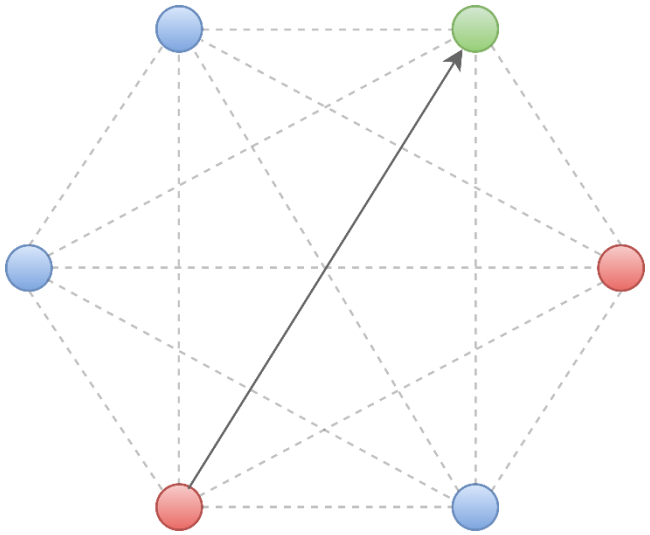

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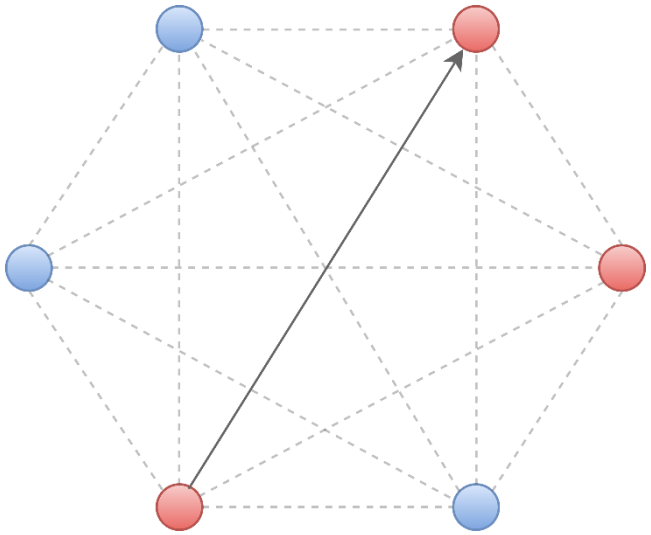

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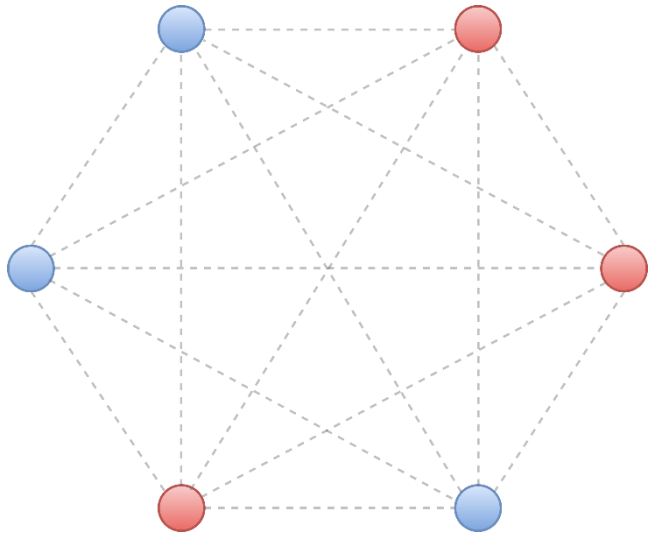

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Related Work

- Approximate Counting: *not much is known*
  - [Michail, PODC, '15] computes an approximate count between *n/2* and *n* in $O(\text{nlogn})$ parallel time w.h.p.
- Leader Election:
  - Simple pairwise-elimination protocol requires linear parallel time [AADFP, Distr. Comp., '06].
  - Any *standard* population protocol requires linear parallel time to solve leader election [DS, DISC, '15].
  - If we strengthen the PP model, we can develop sub-linear time protocols.
    - [AG, ICALP, '15] stabilises in $O(\log^3 n)$ parallel time, assuming $O(\log^3 n)$ states at each agent.
    - [GSU, arXiv, '18] $O(\text{lognloglogn})$ parallel time, assuming $O(\text{loglogn})$ states.
    - [GS, SODA '18] O(log^2n) parallel time, assuming $O(\text{loglogn})$ states.
    - Mediated PP model: [MOKY, Distr. Comp. '12], [DLFSV, TAMC, '17]

Problems and Definitions

Approximate Counting Problem.

We define as *Approximate Counting* the problem in which a leader must determine an estimation $\hat{n}$ of the population size, where $\frac{\hat{n}}{a} < n < \hat{n}$.

Leader Election Problem.

Each node eventually decides whether it is a leader or not, subject to only one node decides that it is the leader.

One-way epidemic.

There is a single node in state *a*, *n-1* nodes in state q and the only effective transition is (a,q)→(a,a). The expected number of steps until all nodes become *a* is Θ(logn) parallel time.

Approximate Counting Protocol

- Requires a unique leader.
- Runtime: $\Theta(\log n)$ parallel time.
- $O(1)$ number of states, except for the unique leader ($O(\log^2 n)$ states).

Protocol 1 Population Size Estimation (PSE)

$Q = \{q, a, l_{c_0,c_1}\}$
δ :

$(l_{0,0}, q) \to (l_{1,0}, a)$
$(a, q) \to (a, a)$
$(l_{c_0,c_1}, q) \to (l_{c_0+1,c_1}, q),\ if\ c_0 > c_1$
$(l_{c_0,c_1}, a) \to (l_{c_0,c_1+1}, a),\ if\ c_0 > c_1$
$(l_{c_0,c_1}, \cdot) \to (halt, \cdot),\ if\ c_0 = c_1$

- The leader is in state *l*, and the rest are in state *q*.
- The leader initiates the epidemic (state *a*).
- The leader stores two counters c_q and c_a.
- Counts the number of interactions with *q* and *a* nodes.
- Termination condition: $c_q = c_a$

Approximate Counting Protocol

THEOREM 1.

Our Approximate Counting protocol gives a constant-factor approximation of $logn$ in $O(logn)$ parallel time w.h.p..

- **First Phase** – Number of infected nodes $[1, \frac{n}{2}]$:
  - c_q reaches $O(logn)$ w. h. p.
  - c_a is increased by a small constant number $(\approx log2)$ *(w.h.p. less that* $(O(\sqrt{logn}))$
  - The protocol does not terminate w.h.p. until more than half of the population has been infected.
- **Second Phase** – Number of infected nodes $(\frac{n}{2}, n]$:
  - c_q is increased by a small constant number $(\approx log2)$ *(w.h.p. less that* $(O(\sqrt{logn}))$
  - c_a eventually catches up c_q.
- 2^{c_q} is an upped bound on the population size.
- First phase: $\Theta(\text{logn})$ parallel time. Second phase: $\Theta(\text{logn})$ parallel time.

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Leader Election Protocol

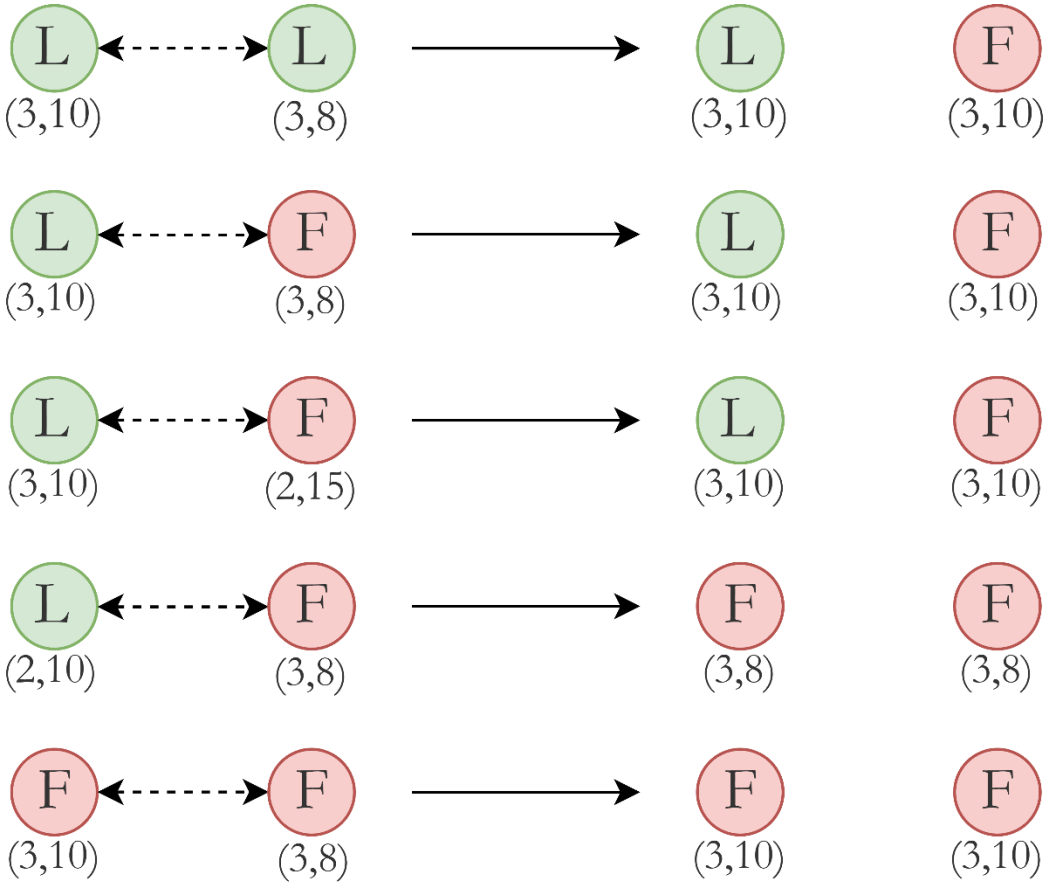
- Assumes that the nodes know an upper bound of $\log(n)$.
- They produce random numbers in $[1, m]$.
- Elects a unique leader in $O(\frac{\log^2 n}{logm})$ parallel time w.h.p..
- Initially all nodes are potential leaders (i.e., all nodes start from the same state).
- The protocol proceeds in rounds by monotonously reducing the set of possible leaders, until only one survives.

- All nodes store three variables: $Round, Random\ Number\ and\ Counter$.
- The *Counter* is increased by one in every interaction, as long as the node remains leader.
- When *Counter* reaches $blogn$, it resets it to zero and increases the $Round$ by one.
- They try to spread their tuple $(Round, Random\ Number)$ throughout the population.

Leader Election Protocol

- We say that a node with the tuple (e_1, r_1) wins during an interaction with (e_2, r_2) if:
  1. $e_1 > e_2$, *or*
  2. $e_1 = e_2$ *and* $r_1 > r_2$

- We call (e_1, r_1) the *dominant tuple* during the round e_1, if it wins every other tuple.



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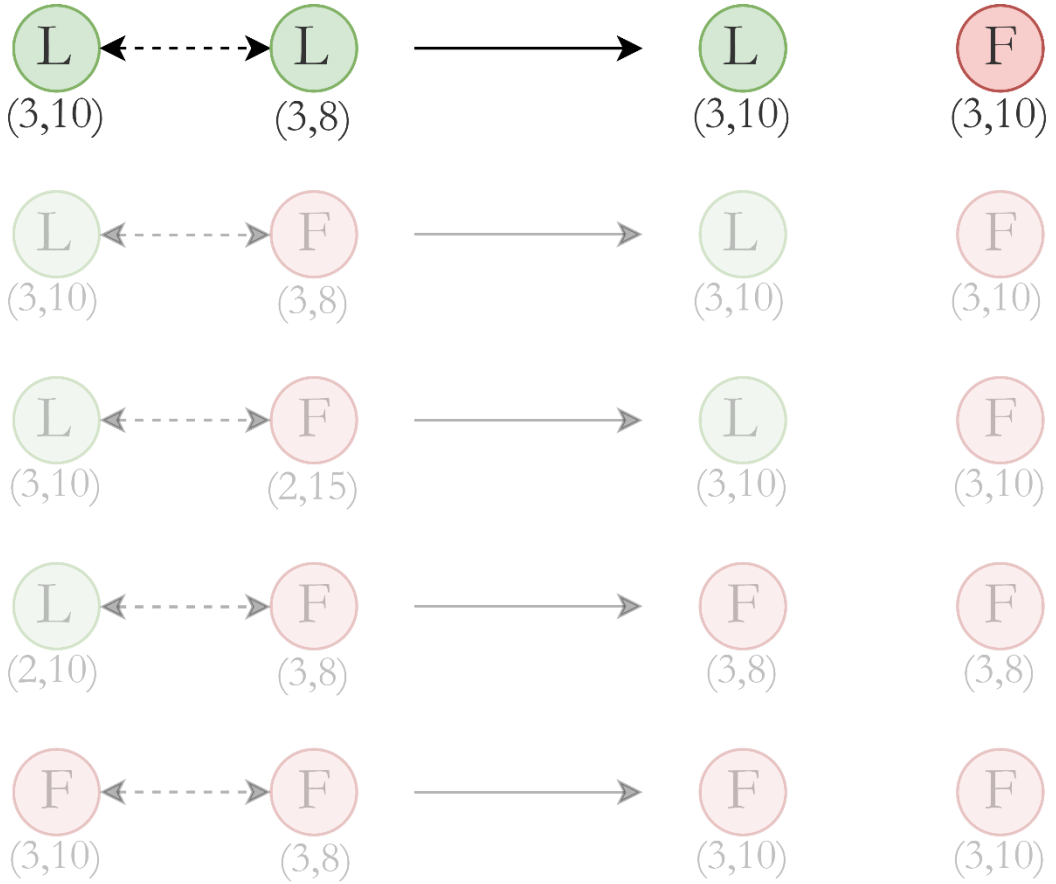

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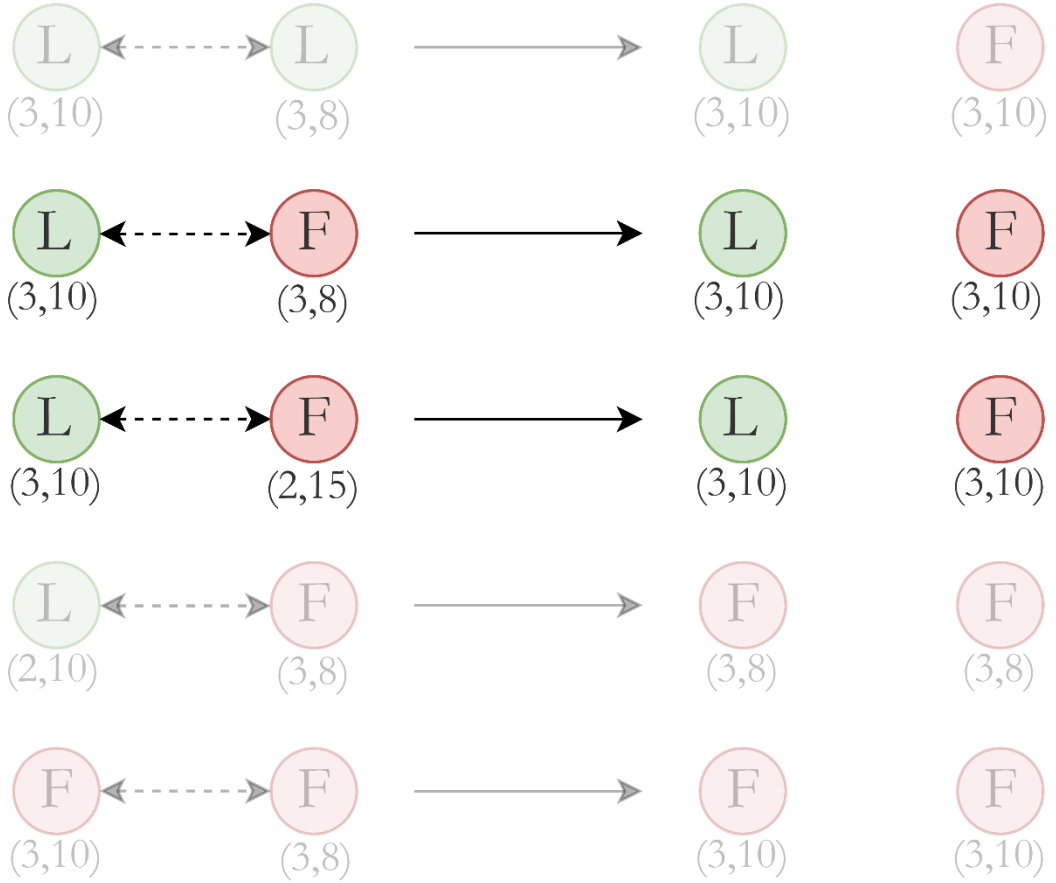

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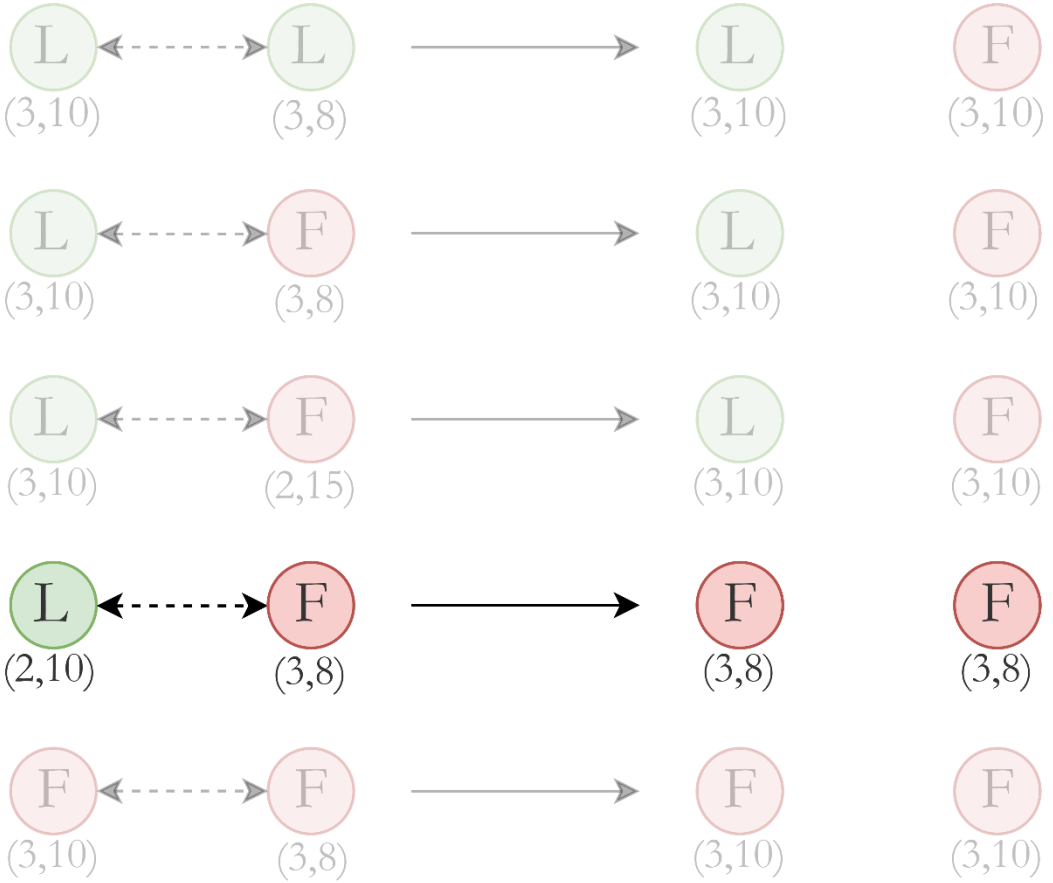

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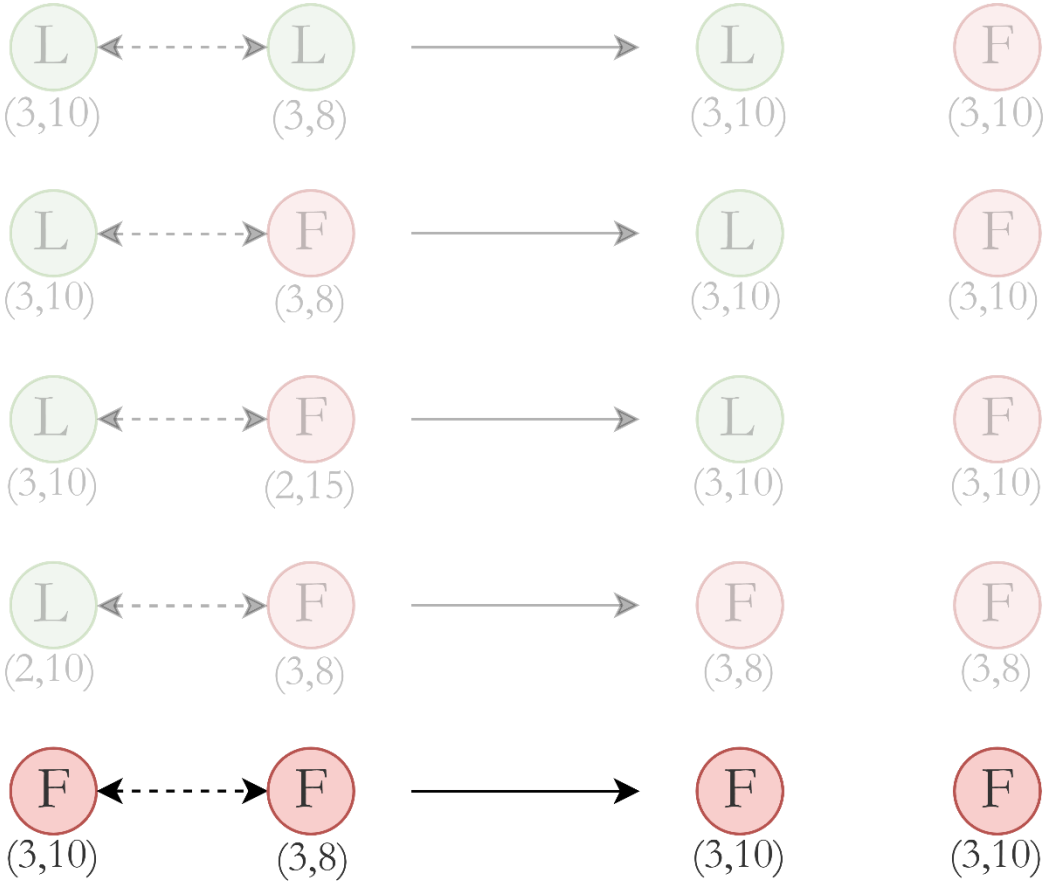

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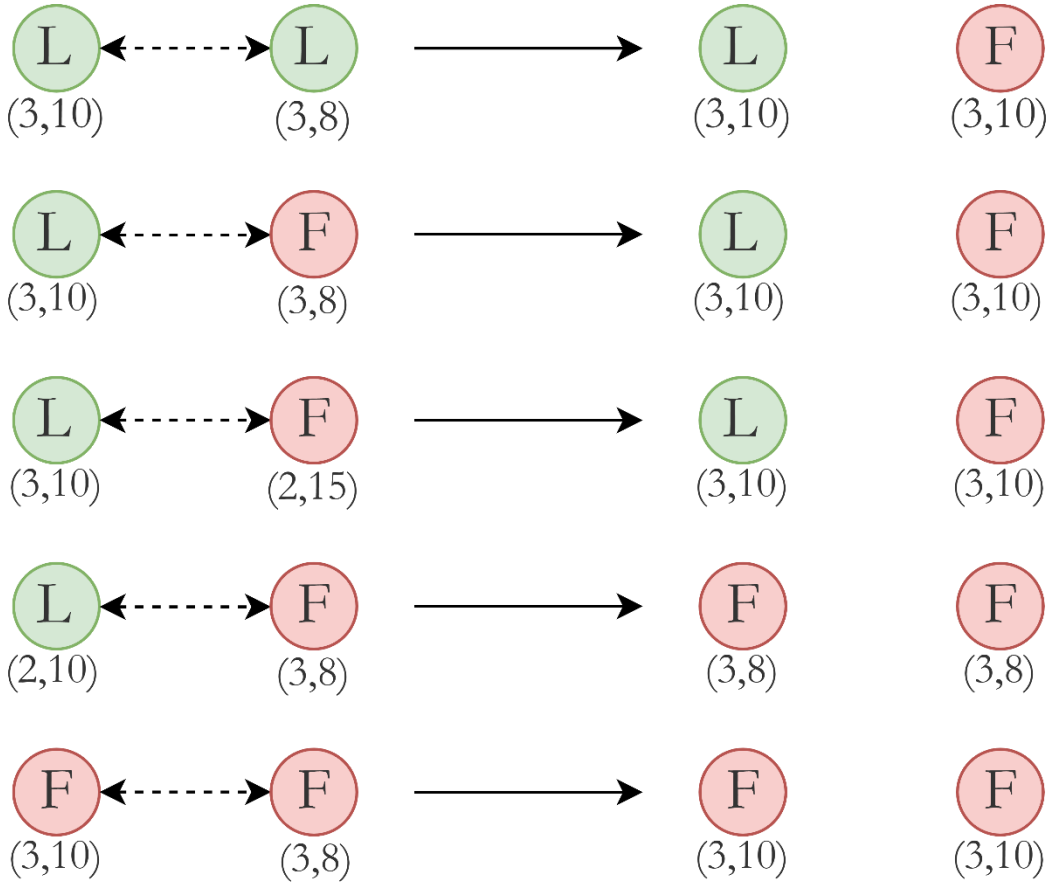

Leader Election Protocol

THEOREM 2.

Our Leader Election protocol elects a unique leader in $O(\frac{\log^2 n}{logm})$ parallel time w.h.p..

- During a round e, the dominant tuple is spread as an epidemic *(in $\Theta(logn)$ parallel time)*.
- No leader can enter the next round if its tuple has not been spread throughout the whole population before.
- At least one leader will always exist in the population.
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Composition of our protocols

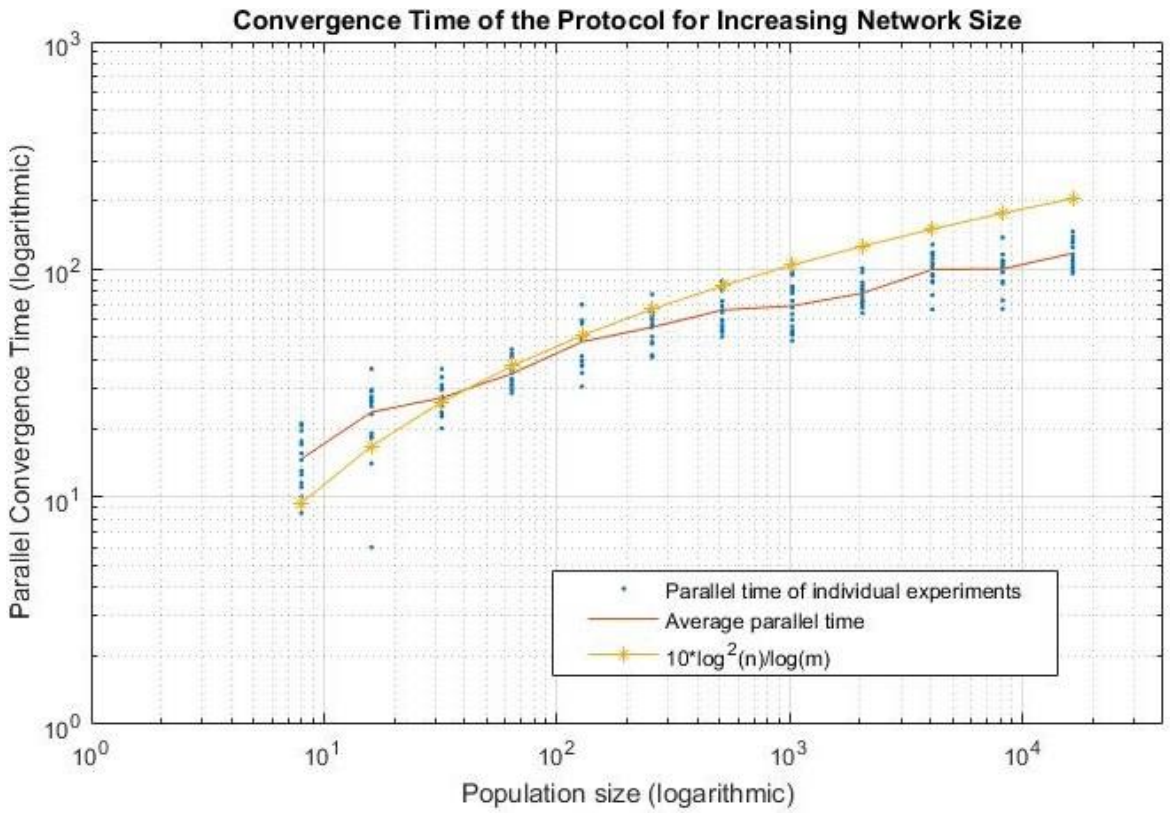
- Our leader election protocol requires a rough estimate on the size of the population
- Our approximate counting protocol requires a unique leader

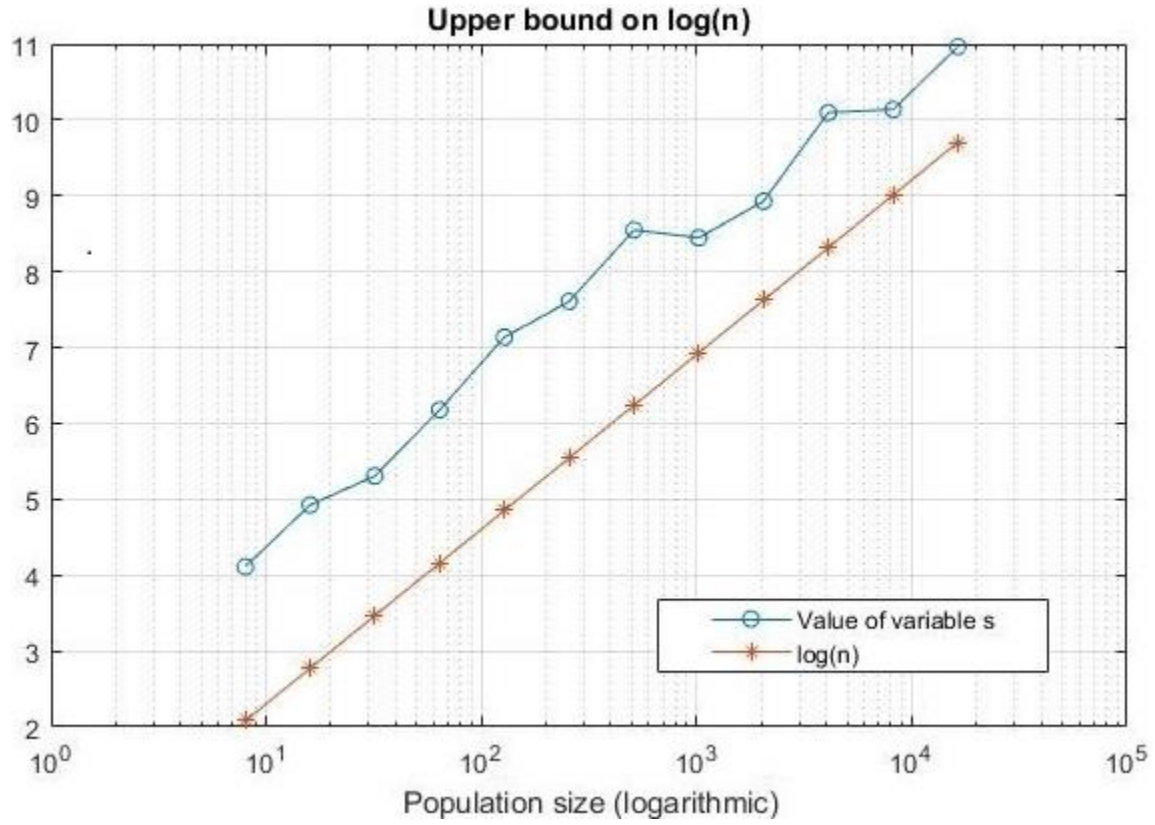
✓ We now combine our protocols in order to construct a leaderless and size-oblivious protocol which elects a unique leader and gives a constant factor approximation of $\log(n)$.

Composition of our protocols

- Leader Election and Approximate Counting protocol:
  - A. In the Leader Election protocol, instead of using one counter, use two counters c_q and c_a
    1. c_q: Counts the non-followers
    2. c_a: Counts the followers
  - B. When $c_q = c_a$
    1. Move to next round
    2. Reset counters ($c_a = 0$ and $c_q = 1$)
    3. Update the value of s as follows: $s' = \frac{s(e_1 - 1) + c_q}{e_1}$
    4. When $e_1 = \left\lceil \frac{as}{logm} \right\rceil$ stop increasing the round.

Composition of our protocols





Conclusions

- Approximate Counting protocol
- Leader Election protocol
- Composition of these protocols

- Can we design a *polylogarithmic* time population protocol that solves the leader election problem, if we allow the agents to communicate only constant amount of bits during an interaction?

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Thank You