# Knights, Knaves, and Logical Reasoning

## Mechanising the Laws of Thought

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The University of Manchester

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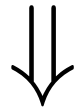
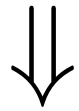
# Introduction

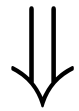
Thinking



Formalising



Modelling



Computing

# A Puzzle

You have decided to take a trip to strange far-off island where the native people have the unusual tradition of dividing themselves into two castes: knights and knaves. Knights are forbidden from ever telling a lie, and knaves are forbidden from ever telling the truth. Both male and female natives are described as knights and knaves.

# A Puzzle

You have decided to take a trip to strange far-off island where the native people have the unusual tradition of dividing themselves into two castes: knights and knaves. Knights are forbidden from ever telling a lie, and knaves are forbidden from ever telling the truth. Both male and female natives are described as knights and knaves. Now, upon your arrival you are greeted by two natives of the island (called A and B) and, eager to learn more about their customs, you ask "Are you knights or knaves?"

# A Puzzle

You have decided to take a trip to strange far-off island where the native people have the unusual tradition of dividing themselves into two castes: knights and knaves. Knights are forbidden from ever telling a lie, and knaves are forbidden from ever telling the truth. Both male and female natives are described as knights and knaves.
Now, upon your arrival you are greeted by two natives of the island (called A and B) and, eager to learn more about their customs, you ask "Are you knights or knaves?"
The first native A replies "At least one of us is a knave". What are you to understand from this?

# Mathematical(?) logic

# Sudoku

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | 7 | | | 4 | 1 | |
| | | 3 | | 2 | | | | 6 |
| 1 | | 7 | 4 | | | 5 | 2 | 3 |
| 4 | | 1 | 6 | | | | 8 | |
| | 2 | 9 | | 7 | | 6 | 3 | |
| | 7 | | | | 4 | 2 | | 1 |
| 7 | 5 | 2 | | | 6 | 3 | | 9 |
| 3 | | | | 4 | | 1 | | |
| | 1 | 4 | | | 3 | | | |

# Sudoku

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | 7 | | | 4 | 1 | |
| | | 3 | | 2 | | | | 6 |
| 1 | | 7 | 4 | | | 5 | 2 | 3 |
| 4 | | 1 | 6 | | | | 8 | |
| | 2 | 9 | | 7 | | 6 | 3 | |
| | 7 | | | | 4 | 2 | | 1 |
| 7 | 5 | 2 | | | 6 | 3 | | 9 |
| 3 | | | | 4 | | 1 | | |
| | 1 | 4 | | | 3 | | | |

**If** there isn't a 7 in this row, **and** there isn't a 7 in this column, **and** there isn't a 7 in this square, **then** you can put a 7 in this box.

# Sudoku

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | 7 | | | 4 | 1 | |
| | | 3 | | 2 | | | | 6 |
| 1 | | 7 | 4 | | | 5 | 2 | 3 |
| 4 | | 1 | 6 | | | | 8 | |
| | 2 | 9 | | 7 | | 6 | 3 | |
| | 7 | | | | 4 | 2 | | 1 |
| 7 | 5 | 2 | | | 6 | 3 | | 9 |
| 3 | | | | 4 | | 1 | | |
| | 1 | 4 | | | 3 | | | |

**If** there isn't a 7 in this row, **and** there isn't a 7 in this column, **and** there isn't a 7 in this square, **then** you can put a 7 in this box.
This box must contain a 7 **or** a 9.

# Propositions

An expression which is either true or false.

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Proposition test: Is it true that. . . ?

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- Manchester
- Grass is green
- We're in Manchester
- What's your name?

# Propositions

An expression which is either true or false.
Proposition test: Is it true that. . . ?

- $2 + 2 = 5$
- Manchester
- Grass is green
- We're in Manchester
- What's your name?
- It's raining

# Not $\neg$

| $p$ | $\neg p$ |
|---|---|
| F | T |
| T | F |

# Not $\neg$

| $p$ | $\neg p$ |
|---|---|
| F | T |
| T | F |

It's *not* raining.

# Not $\neg$

| $p$ | $\neg p$ |
|---|---|
| F | T |
| T | F |

It's *not* raining.

Grass is *not* green.

# And &

| $p$ | $q$ | $p\&q$ |
|---|---|---|
| F | F | F |
| F | T | F |
| T | F | F |
| T | T | T |

# And &

| $p$ | $q$ | $p \& q$ |
|---|---|---|
| F | F | F |
| F | T | F |
| T | F | F |
| T | T | T |

Grass is green *and* it's raining.

# And &

| $p$ | $q$ | $p \& q$ |
|---|---|---|
| F | F | F |
| F | T | F |
| T | F | F |
| T | T | T |

Grass is green *and* it's raining.

We're in Manchester *and* we're in France.

# Or $|$

| $p$ | $q$ | $p \mid q$ |
|---|---|---|
| F | F | F |
| F | T | T |
| T | F | T |
| T | T | T |

# Or $|$

| $p$ | $q$ | $p \mid q$ |
|---|---|---|
| F | F | F |
| F | T | T |
| T | F | T |
| T | T | T |

Take an aspirin *or* lie down.

# Or |

| $p$ | $q$ | $p \mid q$ |
|---|---|---|
| F | F | F |
| F | T | T |
| T | F | T |
| T | T | T |

Take an aspirin *or* lie down.

You can have milk or sugar in your tea.

# Implication - If, then $\rightarrow$

| $p$ | $q$ | $p \rightarrow q$ |
|---|---|---|
| F | F | T |
| F | T | T |
| T | F | F |
| T | T | T |

# Implication - If, then $\rightarrow$

| $p$ | $q$ | $p \rightarrow q$ |
|---|---|---|
| F | F | T |
| F | T | T |
| T | F | F |
| T | T | T |

*If* you get 90% on this assignment, *then* you'll pass the course.

# Implication - If, then $\rightarrow$

| $p$ | $q$ | $p \rightarrow q$ |
|---|---|---|
| F | F | T |
| F | T | T |
| T | F | F |
| T | T | T |

*If* you get 90% on this assignment, *then* you'll pass the course.

*If* you're late, *then* you'll give me a fiver.

# Biimplication - If and only if $\leftrightarrow$

| $p$ | $q$ | $p \leftrightarrow q$ |
|---|---|---|
| F | F | T |
| F | T | F |
| T | F | F |
| T | T | T |

# Biimplication - If and only if $\leftrightarrow$

| $p$ | $q$ | $p \leftrightarrow q$ |
|---|---|---|
| F | F | T |
| F | T | F |
| T | F | F |
| T | T | T |

I'll buy you a new wallet *if (and only if)* you need one.

# Biimplication - If and only if $\leftrightarrow$

| $p$ | $q$ | $p \leftrightarrow q$ |
|---|---|---|
| F | F | T |
| F | T | F |
| T | F | F |
| T | T | T |

I'll buy you a new wallet *if (and only if)* you need one.
He studies *if (and only if)* he can.

# An Example: $(p\&q) \rightarrow r$

| $p$ | $q$ | $r$ | $(p\&q)$ | $(p\&q) \rightarrow r$ |
|---|---|---|---|---|
| F | F | F | | |
| F | F | T | | |
| F | T | F | | |
| F | T | T | | |
| T | F | F | | |
| T | F | T | | |
| T | T | F | | |
| T | T | T | | |

# An Example: $(p\&q) \rightarrow r$

| $p$ | $q$ | $r$ | $(p\&q)$ | $(p\&q) \rightarrow r$ |
|---|---|---|---|---|
| F | F | F | F | |
| F | F | T | F | |
| F | T | F | F | |
| F | T | T | F | |
| T | F | F | F | |
| T | F | T | F | |
| T | T | F | | |
| T | T | T | | |

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| $p$ | $q$ | $r$ | $(p\&q)$ | $(p\&q) \rightarrow r$ |
|---|---|---|---|---|
| F | F | F | F | |
| F | F | T | F | |
| F | T | F | F | |
| F | T | T | F | |
| T | F | F | F | |
| T | F | T | F | |
| T | T | F | T | |
| T | T | T | T | |

# An Example: $(p\&q) \rightarrow r$

| $p$ | $q$ | $r$ | $(p\&q)$ | $(p\&q) \rightarrow r$ |
|---|---|---|---|---|
| F | F | F | F | T |
| F | F | T | F | T |
| F | T | F | F | T |
| F | T | T | F | T |
| T | F | F | F | T |
| T | F | T | F | T |
| T | T | F | T | |
| T | T | T | T | |

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| $p$ | $q$ | $r$ | $(p\&q)$ | $(p\&q) \rightarrow r$ |
|---|---|---|---|---|
| F | F | F | F | T |
| F | F | T | F | T |
| F | T | F | F | T |
| F | T | T | F | T |
| T | F | F | F | T |
| T | F | T | F | T |
| T | T | F | T | F |
| T | T | T | T | |

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| $p$ | $q$ | $r$ | $(p\&q)$ | $(p\&q) \rightarrow r$ |
|---|---|---|---|---|
| F | F | F | F | T |
| F | F | T | F | T |
| F | T | F | F | T |
| F | T | T | F | T |
| T | F | F | F | T |
| T | F | T | F | T |
| T | T | F | T | F |
| T | T | T | T | T |

# The Trick

$k_A$ means A is a knight.

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$k_A$ means A is a knight.
$\neg k_A$ means A is a knave.
"A said X" is the same as $k_A \leftrightarrow X$.
Draw a truth table for this and find a *satisfying assignment*. (A row where the final column contains $T$.)

# A Solution

“At least one of us is a knave.”

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$k_A$ means A is a knight.
$\neg k_A$ means A is a knave.

| $k_A$ | $k_B$ | $\neg k_A$ | $\neg k_B$ | $\neg k_A \vert \neg k_B$ | $k_A \leftrightarrow (\neg k_A \vert \neg k_B)$ |
|---|---|---|---|---|---|
| F | F | T | T | T | |
| F | T | T | F | T | |
| T | F | F | T | T | |
| T | T | F | F | F | |

# A Solution

"At least one of us is a knave."
$k_A$ means A is a knight.
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| $k_A$ | $k_B$ | $\neg k_A$ | $\neg k_B$ | $\neg k_A \vert \neg k_B$ | $k_A \leftrightarrow (\neg k_A \vert \neg k_B)$ |
|---|---|---|---|---|---|
| F | F | T | T | T | F |
| F | T | T | F | T | |
| T | F | F | T | T | |
| T | T | F | F | F | |

# A Solution

"At least one of us is a knave."
$k_A$ means A is a knight.
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| $k_A$ | $k_B$ | $\neg k_A$ | $\neg k_B$ | $\neg k_A \vert \neg k_B$ | $k_A \leftrightarrow (\neg k_A \vert \neg k_B)$ |
|---|---|---|---|---|---|
| F | F | T | T | T | F |
| F | T | T | F | T | F |
| T | F | F | T | T | |
| T | T | F | F | F | |

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"At least one of us is a knave."
$k_A$ means A is a knight.
$\neg k_A$ means A is a knave.

| $k_A$ | $k_B$ | $\neg k_A$ | $\neg k_B$ | $\neg k_A \vert \neg k_B$ | $k_A \leftrightarrow (\neg k_A \vert \neg k_B)$ |
|---|---|---|---|---|---|
| F | F | T | T | T | F |
| F | T | T | F | T | F |
| T | F | F | T | T | T |
| T | T | F | F | F | |

# A Solution

"At least one of us is a knave."
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| $k_A$ | $k_B$ | $\neg k_A$ | $\neg k_B$ | $\neg k_A \vert \neg k_B$ | $k_A \leftrightarrow (\neg k_A \vert \neg k_B)$ |
|---|---|---|---|---|---|
| F | F | T | T | T | F |
| F | T | T | F | T | F |
| T | F | F | T | T | T |
| T | T | F | F | F | F |

# Automating the Process

Truth table

- mechanical
- time consuming ($2^n$ rows!)
- tedious

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Let a computer do it for you!

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Truth table

- mechanical
- time consuming ($2^n$ rows!)
- tedious

Let a computer do it for you!

- ideal for mechanical tasks
- only needs an input formula
- much faster than us
- the output is easily customisable

# From Solving to Modelling

Computers solve the puzzle
(part of the fun is gone)

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Computers solve the puzzle
(part of the fun is gone)

Our contribution is still fundamental!

- finding the right procedure (hopefully a fast one)
- changing focus: Solving $\Rightarrow$ Modelling

A says "At least one of us is a knave."
$k_A \leftrightarrow \neg k_A | \neg k_B$

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Computers solve the puzzle
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Our contribution is still fundamental!

- finding the right procedure (hopefully a fast one)
- changing focus: Solving $\Rightarrow$ Modelling

A says "At least one of us is a knave."
$k_A \leftrightarrow \neg k_A | \neg k_B$

Can be (really) hard, but you only have to do it once!

# Modelling a Sudoku

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | 7 | | | 4 | 1 | |
| | | 3 | | 2 | | | | 6 |
| 1 | | 7 | 4 | | | 5 | 2 | 3 |
| 4 | | 1 | 6 | | | | 8 | |
| | 2 | 9 | | 7 | | 6 | 3 | |
| | 7 | | | | 4 | 2 | | 1 |
| 7 | 5 | 2 | | | 6 | 3 | | 9 |
| 3 | | | | 4 | | 1 | | |
| | 1 | 4 | | | 3 | | | |

What propositions do we need?

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|---|---|---|---|---|---|---|---|---|
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| 4 | | 1 | 6 | | | | 8 | |
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| | 7 | | | | 4 | 2 | | 1 |
| 7 | 5 | 2 | | | 6 | 3 | | 9 |
| 3 | | | | 4 | | 1 | | |
| | 1 | 4 | | | 3 | | | |

What propositions do we need?

Number $n$ is in row $i$ and column $j$

# Modelling a Sudoku

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | 7 | | | 4 | 1 | |
| | | 3 | | 2 | | | | 6 |
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| | 2 | 9 | | 7 | | 6 | 3 | |
| | 7 | | | | 4 | 2 | | 1 |
| 7 | 5 | 2 | | | 6 | 3 | | 9 |
| 3 | | | | 4 | | 1 | | |
| | 1 | 4 | | | 3 | | | |

What propositions do we need?

Number $n$ is in row $i$ and column $j$

- number 7 is in row 1 and column 4
- number 2 is in row 6 and column 7

# Modelling a Sudoku

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | 7 | | | 4 | 1 | |
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| 1 | | 7 | 4 | | | 5 | 2 | 3 |
| 4 | | 1 | 6 | | | | 8 | |
| | 2 | 9 | | 7 | | 6 | 3 | |
| | 7 | | | | 4 | 2 | | 1 |
| 7 | 5 | 2 | | | 6 | 3 | | 9 |
| 3 | | | | 4 | | 1 | | |
| | 1 | 4 | | | 3 | | | |

What propositions do we need?

Number $n$ is in row $i$ and column $j$

- number 7 is in row 1 and column 4
- number 2 is in row 6 and column 7

729 propositions!

# Modelling a Sudoku (cont'd)

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | 7 | | | 4 | 1 | |
| | | 3 | | 2 | | | | 6 |
| 1 | | 7 | 4 | | | 5 | 2 | 3 |
| 4 | | 1 | 6 | | | | 8 | |
| | 2 | 9 | | 7 | | 6 | 3 | |
| | 7 | | | | 4 | 2 | | 1 |
| 7 | 5 | 2 | | | 6 | 3 | | 9 |
| 3 | | | | 4 | | 1 | | |
| | 1 | 4 | | | 3 | | | |

What to model

- at least one number per cell
- at most one number per cell

# Modelling a Sudoku (cont'd)

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | 7 | | | 4 | 1 | |
| | | 3 | | 2 | | | | 6 |
| 1 | | 7 | 4 | | | 5 | 2 | 3 |
| 4 | | 1 | 6 | | | | 8 | |
| | 2 | 9 | | 7 | | 6 | 3 | |
| | 7 | | | | 4 | 2 | | 1 |
| 7 | 5 | 2 | | | 6 | 3 | | 9 |
| 3 | | | | 4 | | 1 | | |
| | 1 | 4 | | | 3 | | | |

What to model

- at least one number per cell
- at most one number per cell
- no number can be repeated in a row

# Modelling a Sudoku (cont'd)

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | 7 | | | 4 | 1 | |
| | | 3 | | 2 | | | | 6 |
| 1 | | 7 | 4 | | | 5 | 2 | 3 |
| 4 | | 1 | 6 | | | | 8 | |
| | 2 | 9 | | 7 | | 6 | 3 | |
| | 7 | | | | 4 | 2 | | 1 |
| 7 | 5 | 2 | | | 6 | 3 | | 9 |
| 3 | | | | 4 | | 1 | | |
| | 1 | 4 | | | 3 | | | |

What to model

- at least one number per cell
- at most one number per cell
- no number can be repeated in a row
- no number can be repeated in a column

# Modelling a Sudoku (cont'd)

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | 7 | | | 4 | 1 | |
| | | 3 | | 2 | | | | 6 |
| 1 | | 7 | 4 | | | 5 | 2 | 3 |
| 4 | | 1 | 6 | | | | 8 | |
| | 2 | 9 | | 7 | | 6 | 3 | |
| | 7 | | | | 4 | 2 | | 1 |
| 7 | 5 | 2 | | | 6 | 3 | | 9 |
| 3 | | | | 4 | | 1 | | |
| | 1 | 4 | | | 3 | | | |

What to model

- at least one number per cell
- at most one number per cell
- no number can be repeated in a row
- no number can be repeated in a column
- no number can be repeated in a region

# Automated Reasoning

Much more than solving puzzles

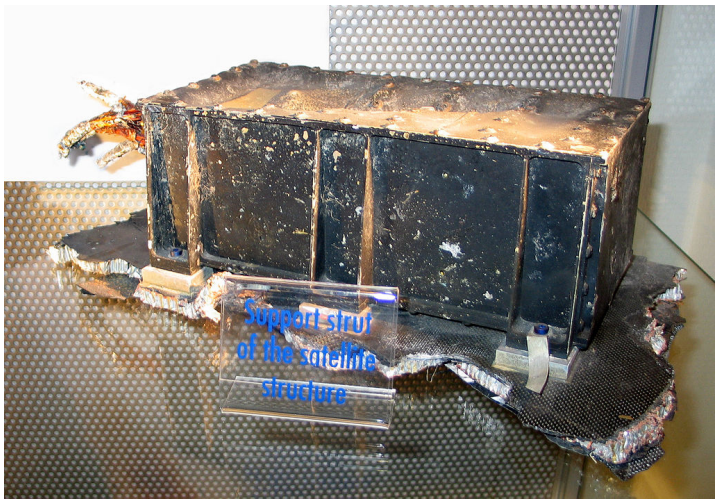
- software and hardware verification
  - Intel and Microsoft
- information management
  - biomedical ontologies, semantic Web, databases
- combinatorial reasoning
  - constraint satisfaction, planning, scheduling
- Internet security
- theorem proving in mathematics

# Where Could Have Been Used

Software bug in Therac-25 a radiation therapy machine led to the death of six patients.

# Where Could Have Been Used (cont'd)

Ariane 5 rocket failure due to a software bug, cost $370 million.



# Automated Reasoning Competitions

- The CADE ATP System Competition (CASC)
- OWL Reasoning Competition (ORE)
- SAT-Race

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- The CADE ATP System Competition (CASC)
- OWL Reasoning Competition (ORE)
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You can bet on the winner!

# Do You Want to Know More?

Look at the references on the handout!