Knights, Knaves, and Logical Reasoning
Mechanising the Laws of Thought

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8 March 2016

Thinking

$\Downarrow$

Formalising

$\Downarrow$

Modelling

$\Downarrow$

Computing

Thinking

A Puzzle

You are on a strange island where people are divided into

- Knights – always saying the truth
- Knaves – always saying lies

You meet two natives of the island Alice and Bob, and ask them

"Are you knights or knaves?"

Alice answers   *"At least one of us is a knave"*

What are Alice and Bob?

Alice: "At least one of us is a knave"

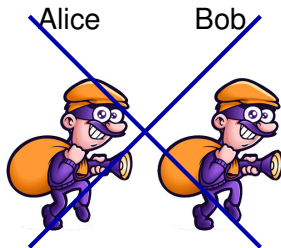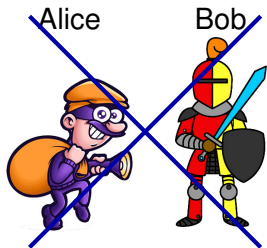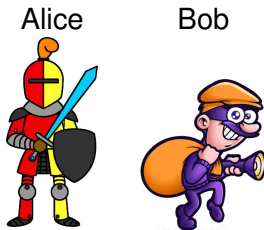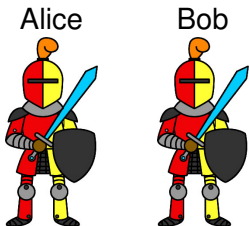# Alice: "At least one of us is a knave"
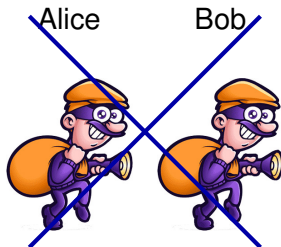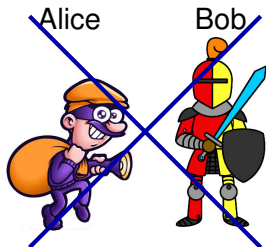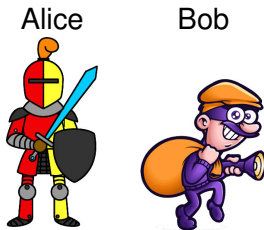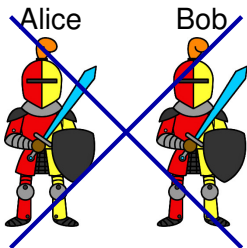
Formalising

Formalising Correct Reasoning

A: Socrates is a man

B: All men are mortal

C: All men are Socrates   C: Socrates is mortal

A: Socrates is a man

B: All men are mortal

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C: Socrates is mortal

Woody Allen - Life and Death

Aristotle

Formalising Correct Reasoning

A: Socrates is a man

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C: Socrates is mortal

Woody Allen - Life and Death

Aristotle

Linguistic, philosophical, or mathematical approaches to formalisation

Today: Propositional Logic

Propositions

An expression which is either true or false.

Propositions

An expression which is either true or false.

Proposition test: Is it true that. . . ?

- $2 + 2 = 5$
- Manchester
- Grass is green
- We're in Manchester
- What's your name?
- It's raining

Not – ¬, And – &

p	$\neg p$
F	T
T	F

It's *not* raining

Grass is *not* green.

Not – ¬, And – &

Not

p	$\neg p$
F	T
T	F

It's *not* raining

Grass is *not* green.

And

p	q	$p \mathbin{\&} q$
F	F	F
F	T	F
T	F	F
T	T	T

Grass is green *and* it's raining.

We're in Manchester *and* we're in France.

Or

p	q	p | q
F	F	F
F	T	T
T	F	T
T	T	T

Take an aspirin *or* lie down.

You can have milk or sugar in your tea.

Or – |, Implication (If, then) – →

Or

p	q	p | q
F	F	F
F	T	T
T	F	T
T	T	T

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Implication

p	q	p → q
F	F	T
F	T	T
T	F	F
T	T	T

If you get 90% on this assignment, *then* you'll pass the course.

If you're late, *then* you'll give me a fiver.

Biimplication (If and only if) – $\leftrightarrow$

Biimplication

p	q	$p \leftrightarrow q$
F	F	T
F	T	F
T	F	F
T	T	T

I'll buy you a new wallet *if (and only if)* you need one.

He studies *if (and only if)* he can.

An Example: $(p \mathbin{\&} q) \rightarrow r$

p	q	r	$(p \mathbin{\&} q)$	$(p \mathbin{\&} q) \rightarrow r$
F	F	F		
F	F	T		
F	T	F		
F	T	T		
T	F	F		
T	F	T		
T	T	F		
T	T	T		

An Example: $(p \,\&\, q) \rightarrow r$

p	q	r	$(p \,\&\, q)$	$(p \,\&\, q) \rightarrow r$
F	F	F	F	
F	F	T	F	
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T	F	F	F	
T	F	T	F	
T	T	F		
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F	F	F	F	T
F	F	T	F	T
F	T	F	F	T
F	T	T	F	T
T	F	F	F	T
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Modelling

k_A = Alice is a knight $\neg k_A$ = Alice is a knave

"Alice says X" is the same as $k_A \leftrightarrow X$.

The Trick

k_A = Alice is a knight $\qquad$ $\neg k_A$ = Alice is a knave

"Alice says X" is the same as $k_A \leftrightarrow X$.

Alice says
- "at least one of us is a knave"
- "I'm a knave or Bob is a knave"
- $\neg k_A \mid \neg k_B$

$$\Rightarrow k_A \leftrightarrow (\neg k_A \mid \neg k_B)$$

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F	F	T	T	T	
F	T	T	F	T	
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From Solving to Modelling

Alice: "At least one of us is a knave"

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The trick: "Alice says X" is the same as $k_A \leftrightarrow X$

"At least one of us is a knave" = $\neg k_A \mid \neg k_B$

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From Solving to Modelling

Alice: "At least one of us is a knave"

k_A = Alice is a knight

The trick: "Alice says X" is the same as $k_A \leftrightarrow X$

"At least one of us is a knave" = $\neg k_A \mid \neg k_B$

Alice says "At least one of us is a knave" = $k_A \leftrightarrow (\neg k_A \mid \neg k_B)$

Can be (really) hard, but you only have to do it once!

			7			4	1	
		3		2				6
1		7	4			5	2	3
4		1	6				8	
	2	9		7		6	3	
	7				4	2		1
7	5	2			6	3		9
3				4		1		
	1	4			3			

What propositions do we need?
Number *n* is in row *i* and column *j*

			7			4	1	
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1		7	4			5	2	3
4		1	6				8	
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	1	4			3			

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Number *n* is in row *i* and column *j*
- number 7 is in row 1 and column 4

729 propositions!
$p_{(1,1,1)}, p_{(1,1,2)}, \ldots, p_{(9,9,9)}$

- $\sim 10^{219}$ rows in a truth table
- $\sim 10^{81}$ atoms in the visible universe

			7			4	1	
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What do we have to model?

- at least one number per cell $(p_{(1,1,1)} \mid p_{(2,1,1)} \mid \ldots \mid p_{(9,1,1)})$
- at most one number per cell $(p_{(1,1,1)} \rightarrow \neg p_{(2,1,1)}, \ldots)$

			7			4	1	
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1		7	4			5	2	3
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- no number can be repeated in a row

Modelling a Sudoku

			7			4	1	
		3		2				6
1		7	4			5	2	3
4		1	6				8	
	2	9		7		6	3	
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- no number can be repeated in a row/column

			7			4	1	
		3		2				6
1		7	4			5	2	3
4		1	6				8	
	2	9		7		6	3	
	7				4	2		1
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What do we have to model?

- at least one number per cell ($p_{(1,1,1)} \mid p_{(2,1,1)} \mid \ldots \mid p_{(9,1,1)}$)
- at most one number per cell ($p_{(1,1,1)} \rightarrow \neg p_{(2,1,1)}, \ldots$)
- no number can be repeated in a row/column/region

Computing

Automating the Process

Truth table

- mechanical
- time consuming (2^n rows!)
- tedious

Automating the Process

Truth table

- mechanical
- time consuming (2^n rows!)
- tedious

Let a computer do it for you!

- ideal for mechanical tasks
- only needs an input formula
- much faster than us
- the output is easily customisable

Automated Reasoning

Much more than solving puzzles!

- software and hardware verification
    - Intel and Microsoft
- information management
    - biomedical ontologies, Semantic Web, databases
- combinatorial reasoning
    - constraint satisfaction, planning, scheduling
- Internet security
- theorem proving in mathematics

Ariane 5 rocket failure due to a software bug, cost $370 million.
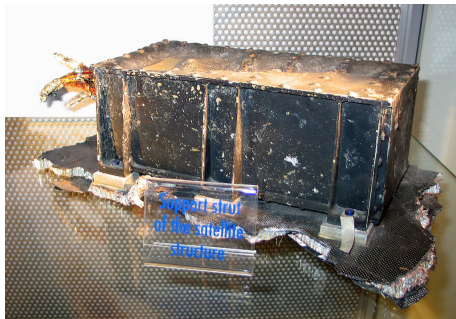
Where Has Been Used

To find and fix a bug in a widely used sorting algorithm!

JDK / JDK-8072909

TimSort fails with ArrayIndexOutOfBoundsException on worst case long arrays

Agile Board   More Actions ▾   ⊙ Views ▾

▾ **Details**

Type:	🐞 Bug		Status:	🔧 Resolved
Priority:	3 P3		Resolution:	Fixed
Affects Version/s:	7u76, 8, 9		Fix Version/s:	9
Component/s:	core-libs			
Labels:	8u60 jdk-port regression webbug			
Subcomponent:	java.util			
Resolved In Build:	b51			
CPU:	x86			
OS:	linux			

▾ **People**

Assignee:	Lev Priima (Inactive)
Reporter:	Webbug Group
👍 Vote (0)	👁 Watch (4)

▾ **Dates**

Created:	2015-02-06 08:22
Updated:	2015-06-09 15:26
Resolved:	2015-02-12 07:44

▾ **Agile**

Where Has Been Used

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Even Amazon and Facebook use automated reasoning techniques!

Automated Reasoning Competitions

- The CADE ATP System Competition (CASC)
- OWL Reasoning Competition (ORE)
- SAT-Race

Feel free to ask questions or look at the references on the handout!