The Power of Counting Steps in Quantitative Games

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Abstract

We study games of infinite duration played on graphs and focus on the strategy complexity of quantitative objectives. Such games are known to admit optimal memoryless strategies over finite graphs, but require infinite-memory strategies in general over infinite graphs.

We provide new lower and upper bounds for the strategy complexity of *mean-payoff* and *total-payoff* objectives over infinite graphs, focusing on whether *step-counter strategies* (sometimes called *Markov strategies*) suffice to implement winning strategies. In particular, we show that over finitely branching arenas, three variants of $\limsup$ mean-payoff and total-payoff objectives admit winning strategies that are based either on a step counter or on a step counter and an additional bit of memory. Conversely, we show that for certain $\liminf$ total-payoff objectives, strategies resorting to a step counter and finite memory are not sufficient. For step-counter strategies, this settles the case of all classical quantitative objectives up to the second level of the Borel hierarchy.

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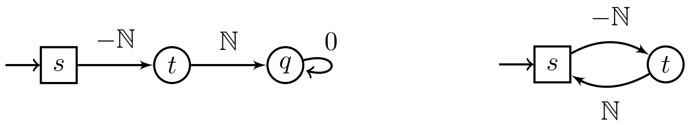
1 Introduction

Two-player (zero-sum, turn-based, perfect-information) games on graphs are an established formalism in formal verification, especially for *reactive synthesis* [1, 14]. They are used to model the interaction between a system, trying to satisfy a given *specification*, against an uncontrollable environment, assumed to act antagonistically as a worst case. We can model the system and its environment as two opposing players, called *Player 1* and *Player 2* respectively, who move a token through the graph of possible system configurations (called the *arena*). The specification is modelled as a winning condition (called *objective* henceforth), which is a set of all those interactions that the system player deems acceptable. The main algorithmic task when using this approach for formal verification is *solving* such games: given an arena, an objective, and an initial vertex, decide whether the system player has a *winning strategy*, which corresponds to a controller for the system that guarantees that the specification holds no matter the behaviour of the environment. Additionally, reactive synthesis aims to *synthesise* (compute a representation of) a winning strategy if one exists.

Strategy complexity To synthesise winning strategies, it is useful to know what kind of resources "suffice", i.e., are needed to implement a winning strategy, should one exist.



(a) The arena $\mathcal{A}_1$. **(b)** The arena $\mathcal{A}_1'$.

Figure 1 Arenas implementing the "match the number" game. Circles designate vertices controlled by Player 1 and squares designate Player 2. The edge labels indicate that for every $i \in \mathbb{N}$ there is a distinct edge with weight $-i$ from s to t, and $+i$ from t to q or from t to s. For $\mathcal{A}_1$, consider the objective "sum of weights exceeds 0". Player 1 can always match and thus win, but needs unbounded memory. The arena $\mathcal{A}_1'$ shows a repeated version for the lim sup *mean*-payoff objective.

This naturally depends on the model used for the interaction (the size and topology of the arena) and on the specification (the type of objective and whether probabilistic or absolute guarantees are required). We assume that strategies make decisions based on some internal memory, that stores and updates an abstraction of the past play.

The simplest strategies are those that are *memoryless*, meaning they base their decisions solely on the current arena vertex. Games on finite arenas where memoryless strategies are sufficient to win can usually be solved in $\mathsf{NP} \cap \mathsf{coNP}$ [29] and winning strategies effectively synthesised. This is true for *parity*, *discounted-payoff* [32], *mean-payoff* [12], and *total-payoff* [8, 17] objectives. Even beyond finite graphs, memoryless strategies may suffice in more general contexts, such as for parity objectives over arenas of arbitrary cardinality [13, 34], or *discounted-payoff objectives* over finitely branching arenas [27, Corollary 2.1].[1] For concurrent (stochastic) *reachability* games on finite arenas, memoryless strategies also suffice [2, 22].

Generally more powerful than memoryless strategies are *finite-memory* strategies, which refer to strategies that can be implemented with a finite-state (Mealy) machine. A canonical class of languages over infinite words, and standard for defining objectives in games, are the ω-regular languages [31, 18]. One of the celebrated related results about reactive synthesis is the *finite-memory determinacy* of ω-regular games [6, 31, 19], which means that if there is a winning strategy in a game on a finite arena and with an ω-regular objective, there is one that can be implemented with a simple finite-state machine (whose size can be bounded). This implies that games with ω-regular objectives can be solved and strategies synthesised, as it bounds the search space for winning strategies. Remarkably, the existence of winning finite-memory strategies for ω-regular games even holds over arbitrary infinite arenas [34]. When finite-memory strategies are sufficient, one of the main questions is usually to *minimise* their size, i.e., to find winning strategies with as few memory states as possible [11, 7, 9, 4, 3].

Already very simple games require infinite memory to win. This especially holds for quantitative objectives, which ask that the aggregate of individual edge weights along a play exceeds some threshold. For instance, consider a game where the environment picks a number and then the controller has to pick a larger one (see Figure 1a). In order to win, Player 1 has to remember the (per se unbounded) initial challenge and no finite memory structure would be sufficient to do so. This objective is not ω-regular as it is built upon an infinite alphabet. We seek to understand for different classes of games, what kind of infinitememory structures are sufficient for winning strategies.

A natural, arguably the simplest, type of infinite memory structure is a *step counter*: it

[1] Thus we consider the strategy complexity in discounted-payoff games as settled for the setting we consider. On infinitely branching arenas, step-counter strategies are insufficient (see Figure 1a).

only remembers how many steps have elapsed since the start of the game. The availability of such a counter is a reasonable assumption for practical applications, as most embedded devices have access to the current time, which suffices when each step takes a fixed amount of time. A *step-counter strategy* is one that, in addition to the current arena vertex, has access to the number of steps elapsed. Notice that in the game in Figure 1a, a step counter does not provide any relevant information (every path to vertex t has length one). Therefore, step-counter strategies do not suffice for Player 1. An important ingredient for these counterexamples is that the underlying arena is infinitely branching (and uses arbitrary weights). For many classes of games on *finitely* branching arenas, strategies based on a step counter and additional finite memory are close to being the simplest kinds of strategies sufficient to win. Examples are especially prevalent in stochastic games. For instance, in the "Big Match" (a concurrent mean-payoff game on a finite arena), neither a step counter nor finite memory is sufficient to play ε-optimally, yet a step counter *together with* one bit is [20]. The same is true for the "Bad Match", which can be presented as a Büchi (repeated reachability) game [24, 33, 23]. This upper bound holds generally for concurrent Büchi games on finite arenas [23].

Quantitative objectives Objectives based on numerical weights are commonly called *quantitative objectives*. These are defined using *quantitative payoff functions*, which combine any finite sequence of weights into an aggregate number. The three more common ones are the discounted-payoff [32], mean-payoff [16, 12], and total-payoff functions [15, 8]. Every payoff function induces four variants of objectives, depending on whether we consider the $\limsup$ or $\liminf$, and on whether we ask that the limit is larger or strictly larger than a threshold. Over infinite arenas, the four variants are not equivalent and infinite-memory strategies are needed for at least one of the players (see [30, Example 8.10.2] and [28]).

To study the strategy complexity for different quantitative objectives, we classify them according to which level of the *Borel hierarchy* they belong to (which also ensures that the games we consider are determined [25]). In the first level of the hierarchy lie the *open* and *closed* objectives (i.e., the sets respectively in $\mathbf{\Sigma}^0_1$ and $\mathbf{\Pi}^0_1$), for which there exist recent characterisations of the sufficient memory structures over finite or infinite arenas [9, 4]. We build on this to establish upper bounds for more complex objectives. All variants of mean-payoff and total-payoff objectives are on the second or third level of the Borel hierarchy. Ohlmann and Skrzypczak [28] study objectives through their topological properties and provide a characterisation of the *prefix-independent* $\mathbf{\Sigma}^0_2$ objectives for which memoryless strategies suffice for Player 1 over arbitrary arenas. It shows in particular that memoryless strategies suffice for Player 1 for the quantitative objectives $\underline{\mathsf{MP}}_{>0}$ and $\underline{\mathsf{TP}}_{>-\infty}$, even over infinitely branching arenas. Over stochastic games, quantitative (in particular $\liminf$ mean-payoff) objectives on infinite arenas generally do not have (ε-)optimal strategies based on a step counter, even for finitely branching Markov decision processes [26].

Our contributions We settle the strategy complexity over infinite, deterministic games for the mean- and total-payoff objectives up to the second level of the Borel hierarchy. In particular, we show for which of these, step-counter strategies are sufficient for Player 1. Our upper bounds all allow for arenas with arbitrary weights, while our strongest lower bounds only use weights -1, 0, and 1. Our results are as follows and summarised in Figure 2.

- For $\underline{\mathsf{TP}}_{>0}$ and $\underline{\mathsf{TP}}_{\geq 0}$, strategies using a step counter and an arbitrary amount of finite memory do not suffice, even over acyclic finitely branching arenas (Theorem 10, Section 3). The proof rules out finite-memory structures using an application of the *infinite Ramsey theorem* to allow Player 2 to stay winning in a particular infinite arena regardless of the

Obj.	Description	Class	Strategy complexity
$\mathsf{MP}_{>0}$	$\bigcup_{m\geq 1}\bigcup_{i\geq 1}\bigcap_{j\geq i}\{w \mid \mathsf{MP}(w_{\leq j}) \geq \frac{1}{m}\}$	$\mathbf{\Sigma}_2^0$	Memoryless (even over infinitely branching arenas) [28]
$\mathsf{TP}_{>-\infty}$	$\bigcup_{m\geq 1}\bigcup_{i\geq 1}\bigcap_{j\geq i}\{w \mid \mathsf{TP}(w_{\leq j}) \geq -m\}$	$\mathbf{\Sigma}_2^0$	
$\mathsf{TP}_{>0}$	$\bigcup_{m\geq 1}\bigcup_{i\geq 1}\bigcap_{j\geq i}\{w \mid \mathsf{TP}(w_{\leq j}) \geq \frac{1}{m}\}$	$\mathbf{\Sigma}_2^0$	SC + FM insufficient (Theorem 10)
$\overline{\mathsf{MP}}_{\geq 0}$	$\bigcap_{m\geq 1}\bigcap_{i\geq 1}\bigcup_{j\geq i}\{w \mid \mathsf{MP}(w_{\leq j}) \geq \frac{-1}{m}\}$	$\mathbf{\Pi}_2^0$	SC sufficient (Corollary 17) FM insufficient (Lemma 4)
$\overline{\mathsf{TP}}_{=+\infty}$	$\bigcap_{m\geq 1}\bigcap_{i\geq 1}\bigcup_{j\geq i}\{w \mid \mathsf{TP}(w_{\leq j}) \geq m\}$	$\mathbf{\Pi}_2^0$	
$\overline{\mathsf{TP}}_{\geq 0}$	$\bigcap_{m\geq 1}\bigcap_{i\geq 1}\bigcup_{j\geq i}\{w \mid \mathsf{TP}(w_{\leq j}) \geq \frac{-1}{m}\}$	$\mathbf{\Pi}_2^0$	SC + 1-bit sufficient (Theorem 20) FM insufficient (Lemma 4) SC insufficient (Lemma 5)

Figure 2 Results for quantitative objectives up to the second level of the Borel hierarchy for finitely branching arenas. *SC* refers to *step counter*, and *FM* refers to *finite memory*.

finite-memory structure of Player 1.

- In Section 5, we provide a sufficient condition for when step-counter strategies suffice over finitely branching arenas for prefix-independent objectives in $\mathbf{\Pi}_2^0$, i.e, countable intersections of open sub-objectives (Theorem 16). This implies in particular that step-counter strategies do suffice for $\overline{\mathsf{MP}}_{\geq 0}$ and $\overline{\mathsf{TP}}_{=+\infty}$ (Corollary 17), which is tight in the sense that finite-memory strategies do not suffice for these objectives, even over acyclic finitely branching arenas (Lemma 4). The proof uses carefully constructed expanding "bubbles", so that within each consecutive bubble, Player 1 can satisfy the next open sub-objective. The step counter is used to determine the current bubble.
- In Section 6, we show that for $\overline{\mathsf{TP}}_{\geq 0}$, which is not prefix-independent, strategies using a step-counter and one additional bit of memory suffice (Theorem 20). This is tight in that neither finite-memory strategies nor step-counter strategies suffice, even over acyclic finitely branching arenas (Lemmas 4 and 5). The proof similarly employs bubbles, but an additional bit is needed to keep track of whether a "sub-objective" has been achieved in the current bubble and then switches to stay in the winning region.

2 Preliminaries

Given a set X, we write X^* for the set of finite words on X, X^+ for the set of non-empty finite words on X, and X^ω for the set of infinite words on X. For $w \in X^*$, we write $|w|$ for the length of w. For $w \in X^\omega$ and $j \in \mathbb{N}$, we write $w_{\leq j}$ for the finite prefix of length j of w.

Games We study two-player zero-sum *games*, each given by an *arena* and an *objective*, as defined below. We refer to the two opposing players as Player 1 and Player 2.

An *arena* is a directed graph with two kinds of vertices where edges are labelled by an element of C, a non-empty set of *colours*. Formally, an arena is a tuple $\mathcal{A} = (V, V_1, V_2, E)$ where $V = V_1 \cup V_2$ is a non-empty set of *vertices*, V_1 and V_2 are disjoint, and $E \subseteq V \times C \times V$ is a set of labelled *edges*. Vertices in V_1 and V_2 are respectively controlled by Player 1 and Player 2, which will appear clearly when we define strategies below. We require that for every vertex $v \in V$, there is an edge $(v, c, v') \in E$ (arenas are "non-blocking"). For $e = (v, c, v')$, we write $\mathsf{from}(e)$ for v, $\mathsf{col}(e)$ for c, and $\mathsf{to}(e)$ for v'. An arena is *finite* if V is finite, and *finitely branching* if for every $v \in V$, the set $\{e \in E \mid \mathsf{from}(e) = v\}$ is finite.

A *history* is a finite sequence $h = e_1 \dots e_n \in E^*$ of edges such that for $i \in \{1, \dots, n-1\}$, $\mathsf{to}(e_i) = \mathsf{from}(e_{i+1})$. We write $\mathsf{from}(h)$ for $\mathsf{from}(e_1)$, $\mathsf{to}(h)$ for $\mathsf{to}(e_n)$, and $\mathsf{col}(h)$ for the sequence $\mathsf{col}(e_1)\dots\mathsf{col}(e_n) \in C^*$. For convenience, we assume that for every vertex v, there is a distinct *empty history* λ_v such that $\mathsf{from}(\lambda_v) = \mathsf{to}(\lambda_v) = v$. The set of histories of $\mathcal{A}$ is denoted as $\mathsf{hists}(\mathcal{A})$. For $p \in \{1, 2\}$, we write $\mathsf{hists}_p(\mathcal{A})$ for the set of histories h such that $\mathsf{to}(h) \in V_p$. A *play* is an infinite sequence of edges $\rho = e_1e_2\dots \in E^\omega$ such that for $i \geq 1$, $\mathsf{to}(e_i) = \mathsf{from}(e_{i+1})$. We write $\mathsf{from}(\rho)$ for $\mathsf{from}(e_1)$ and $\mathsf{col}(\rho)$ for $\mathsf{col}(e_1)\mathsf{col}(e_2)\dots \in C^\omega$. A history h (resp. a play ρ) is said to be *from* v if $v = \mathsf{from}(h)$ (resp. $v = \mathsf{from}(\rho)$).

An *objective* (sometimes called a *winning condition* in the literature) is a set $O \subseteq C^\omega$. An objective O is *prefix-independent* if for all $w \in C^*$, $w' \in C^\omega$, $ww' \in O$ if and only if $w' \in O$.

Strategies A *strategy of Player* p *on* $\mathcal{A}$ is a function $\sigma\colon \mathsf{hists}_p(\mathcal{A}) \to E$. A play $\rho = e_1e_2\dots$ is *consistent with a strategy* σ *of Player* p if for all finite prefixes h of ρ such that $\mathsf{to}(h) \in V_p$, $\sigma(h) = e_{|h|+1}$. A strategy σ of Player 1 is *winning for objective* O *from a vertex* v if all plays from v consistent with σ induce a sequence of colours in O. For a fixed objective, the set of vertices of an arena $\mathcal{A}$ from which a winning strategy for Player 1 exists is called the *winning region of Player 1 on* $\mathcal{A}$ and is denoted $W_{\mathcal{A},1}$. A strategy σ of Player 1 is *uniformly winning for objective* O *in* $\mathcal{A}$ if σ is winning from every vertex of the winning region of $\mathcal{A}$.

A *memory structure for an arena* $\mathcal{A} = (V, V_1, V_2, E)$ is a tuple $\mathcal{M} = (M, m_0, \delta)$ where M is a set of *memory states*, $m_0 \in M$ is an *initial state*, and $\delta\colon M \times E \to M$ is a *memory update function*. We extend δ to a function $\delta^*\colon M \times E^* \to M$ in a natural way. A memory structure $\mathcal{M}$ is *finite* if M is finite. A strategy σ of Player p on $\mathcal{A}$ is *based on* $\mathcal{M}$ if there exists a function $f\colon V_p \times M \to E$ such that, for all $h \in \mathsf{hists}_p(\mathcal{A})$, $\sigma(h) = f(\mathsf{to}(h), \delta^*(m_0, h))$. We will abusively assume that a strategy based on a memory structure is this function f. For two memory structures $\mathcal{M}$ and $\mathcal{M}'$, we denote their *direct product* by $\mathcal{M} \times \mathcal{M}'$.

A *memoryless strategy* is a strategy based on a memory structure with a single memory state. A 1-*bit strategy* is a strategy based on a memory structure with two memory states. A *step counter* is a memory structure $\mathcal{S} = (\mathbb{N}, 0, (s, e) \mapsto s + 1)$ that simply counts the number of steps already elapsed in a game. A strategy σ of Player p on $\mathcal{A}$ is a *step-counter strategy* if σ is based on a step counter; in other words, if there is a function $f\colon V_p \times \mathbb{N} \to E$ such that $\sigma(h) = f(\mathsf{to}(h), |h|)$. This means that σ only considers the current vertex and the number of steps elapsed to make its decisions. Step-counter strategies are sometimes called "Markov strategies" [33, 21]. A *step-counter +* 1-*bit strategy* is based on the direct product of a step counter and a memory structure with two states. A *step-counter and finite-memory strategy* is based on the direct product of a step counter and a finite memory structure.

We say that a kind of strategies *suffices for objective* O *over a class of arenas* if, for all arenas in this class, from all vertices of her winning region, Player 1 has a winning strategy of this kind. We say that a kind of strategies *suffices uniformly for objective* O *over a class of arenas* if, for all arenas in this class, Player 1 has a uniformly winning strategy of this kind.

For an arena $\mathcal{A} = (V, V_1, V_2, E)$ and a memory structure $\mathcal{M} = (M, m_0, \delta)$, we write $\mathcal{A} \otimes \mathcal{M}$ for the *product between* $\mathcal{A}$ *and* $\mathcal{M}$. It is the arena (V', V_1', V_2', E') such that $V' = V \times M$, $V_1' = V_1 \times M$, $V_2' = V_2 \times M$, and $E' = \{((v, m), c, (v', \delta(m, c))) \mid (v, c, v') \in E, m \in M\}$. Observe that Player 1 has a winning strategy based on $\mathcal{M}$ from a vertex v in an arena $\mathcal{A}$ if and only if Player 1 has a winning memoryless strategy from vertex (v, m_0) in $\mathcal{A} \otimes \mathcal{M}$.

To simplify reasonings over specific arenas, we show that step counters do not have any use when the arena already *encodes the step count*.

▶ **Lemma 1.** *Let* $\mathcal{A} = (V, V_1, V_2, E)$ *be an arena, and* $v_0 \in V$ *be an initial vertex. Assume that for each pair of histories* h_1, h_2 *from* v_0 *to some* $v \in V$*, we have* $|h_1| = |h_2|$ *(i.e., the arena already "encodes the step count from* v_0*"). Then, strategies based on the product of a step counter and a memory structure* $\mathcal{M}$ *can be simulated from* v_0 *by strategies based on* $\mathcal{M}$*.*

Proof. Let $\mathcal{M} = (M, m_0, \delta)$. By hypothesis on $\mathcal{A}$, there exists $n_v \in \mathbb{N}$ the length of any history from v_0 to v. A step-counter-free strategy $\sigma\colon V_1 \times M \to E$ can be built from $\sigma'\colon V_1 \times \mathbb{N} \times M \to E$ (which depends on the step), by defining $\sigma(v, m)$ as $\sigma'(v, n_v, m)$. ◀

Quantitative objectives We consider classical quantitative objectives: mean-payoff and total-payoff objectives, as defined below. Let $C \subseteq \mathbb{Q}$ (when colours are rational numbers, we often refer to them as *weights*). For a finite word $w = c_1 \ldots c_{|w|} \in C^*$, define $\mathsf{TP}(w) = \sum_{i=1}^{|w|} c_i$ for the *total payoff* of the word, i.e., the sum of the weights it contains. Further, when $|w| \geq 1$, let $\mathsf{MP}(w) = \mathsf{TP}(w)/|w|$ denote the *mean payoff* of the word w, i.e., the mean of the weights it contains. We extend any such aggregate function $X\colon C^* \to \mathbb{R}$ to infinite words by taking limits: for $w \in C^\omega$, we define $\overline{X}(w) = \limsup_j X(w_{\leq j})$ and $\underline{X}(w) = \liminf_j X(w_{\leq j})$. Fixing a binary relation $\triangleright \subseteq \mathbb{R}^2$ and threshold $r \in \mathbb{Q} \cup \{-\infty, \infty\}$, this naturally defines objectives $\overline{X}_{\triangleright r} = \{w \in C^\omega \mid \overline{X}(w) \triangleright r\}$ and $\underline{X}_{\triangleright r} = \{w \in C^\omega \mid \underline{X}(w) \triangleright r\}$.

In particular, we are interested in the limit infimum/supremum objectives for total and mean payoff.[2] We consider the mean-payoff variants with threshold $r \in \mathbb{Q}$, and the total-payoff variants with $r \in \mathbb{Q} \cup \{-\infty, +\infty\}$. Note that all four mean-payoff objectives and all four total-payoff objectives with ∞ threshold are prefix-independent, but the four total-payoff objectives with threshold in $\mathbb{Q}$ are not prefix-independent.

▶ Remark 2. Our results are generally stated for threshold $r = 0$. This is without loss of generality, as the results deal with large classes of arenas, and little modifications to the arenas allow to reduce from an arbitrary rational threshold to threshold 0. ⌟

Topology of objectives For $w \in C^*$, we write $wC^\omega = \{ww' \mid w' \in C^\omega\}$ for the objective containing all infinite words that start with w (it is sometimes called the *cylinder* or *cone* of w). An objective O is *open* if there is a set $A \subseteq C^*$ such that $O = \bigcup_{w \in A} wC^\omega$. For an open objective O, we say that a finite word $w \in C^*$ *already satisfies* O if $wC^\omega \subseteq O$. If an objective is open, then by definition, any infinite word it contains has a finite prefix that already satisfies it. An objective is *closed* if it is the complement of an open set.

Open and closed objectives are at the first level of the *Borel hierarchy*; the set of open (resp. closed) objectives is denoted $\mathbf{\Sigma}_1^0$ (resp. $\mathbf{\Pi}_1^0$). For $i > 1$, we can define $\mathbf{\Sigma}_i^0$ as all the countable unions of sets in $\mathbf{\Pi}_{i-1}^0$, and $\mathbf{\Pi}_i^0$ as all the countable intersections of sets in $\mathbf{\Sigma}_{i-1}^0$. All the objectives considered in this paper lie in the first three levels of this hierarchy, and we focus on those on the second level.

3 Lower bounds

We provide lower bounds on the size/structure of the memory to build winning strategies, focusing on objectives $\overline{\mathsf{MP}}_{\geq 0}$, $\overline{\mathsf{TP}}_{=+\infty}$, $\overline{\mathsf{TP}}_{\geq 0}$, and $\underline{\mathsf{TP}}_{>0}$, which are the four objectives on the second level of Borel hierarchy for which we want to establish whether step-counters strategies suffice. We mention where our constructions directly work for further objectives.

All lower bounds are based on the simple idea that one player chooses some number and the other must match it. We first observe that on infinitely branching arenas with arbitrary weights, neither finite memory nor a step counter, nor both together, is sufficient. The proof uses the arenas from Figure 1, discussed informally in Section 1 (full proofs in Appendix A).

[2] We only consider objectives where the threshold is a *lower* bound ($\triangleright \in \{>, \geq\}$); each variant with *upper* bound behaves like a variant with lower bound when we replace each weight c in arenas with its additive inverse $-c$ and switch the sup/inf (for instance, $\overline{\mathsf{MP}}_{<r}$ behaves like $\underline{\mathsf{MP}}_{>r}$ when we invert the weights).

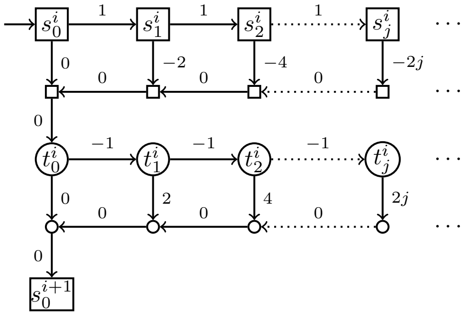


Figure 3 The arena $\mathcal{A}_2$ is acyclic and every vertex has finite in- and out-degree.

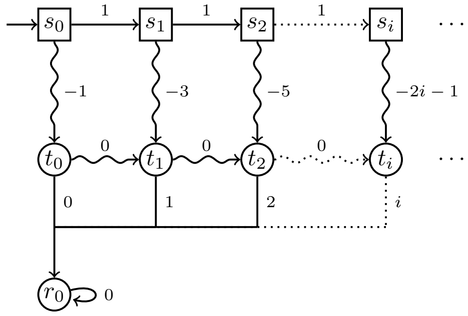


Figure 4 The arena $\mathcal{A}_3$. Arrows $s_i \overset{-2i-1}{\rightsquigarrow} t_i$ are shorthand for paths of length $2i+1$ with edge weights -1, and $t_i \overset{0}{\rightsquigarrow} t_{i+1}$ are shorthand for paths of length 3 with edge weights 0.

▶ **Lemma 3.** *Over infinitely branching arenas with arbitrary weights, strategies based on a step counter and finite memory are not sufficient for Player 1 for objectives* $\overline{\mathsf{MP}}_{>0}$, $\overline{\mathsf{MP}}_{\geq 0}$, $\overline{\mathsf{TP}}_{=+\infty}$, $\underline{\mathsf{TP}}_{>0}$, $\underline{\mathsf{TP}}_{\geq 0}$, $\overline{\mathsf{TP}}_{>0}$ *and* $\overline{\mathsf{TP}}_{\geq 0}$.

We now establish lower bounds over finitely branching arenas. Firstly, the example $\mathcal{A}_1'$ can be made finitely branching and acyclic, as depicted in Figure 3. The resulting arena, $\mathcal{A}_2$, simply unfolds $\mathcal{A}_1'$ so that any edge $(s, -j, t)$ is replaced by a finite path $s_0^i \to \cdots \to s_j^i \to t_0^i$, and similarly for the responses. This construction works as long as one can discourage (i.e., make losing) the choice to stay on the infinite intermediate chain of vertices and not moving on to a vertex controlled by the opponent. Here, this is achieved by using weights 1 on the chains of Player 2 and weights -1 on the chains of Player 1, which are then compensated by weights twice as large. In practice, edges with weights $i \in \mathbb{N}$ (resp. $-i \in -\mathbb{N}$) can be replaced by chains of i weights 1 (resp. i weights -1). This allows to obtain lower bounds on the $\limsup$ objectives.

▶ **Lemma 4.** *Over finitely branching arenas, finite-memory strategies are not sufficient for Player 1 for objectives* $\overline{\mathsf{MP}}_{>0}$, $\overline{\mathsf{MP}}_{\geq 0}$, $\overline{\mathsf{TP}}_{=+\infty}$, $\overline{\mathsf{TP}}_{>0}$, *and* $\overline{\mathsf{TP}}_{\geq 0}$.

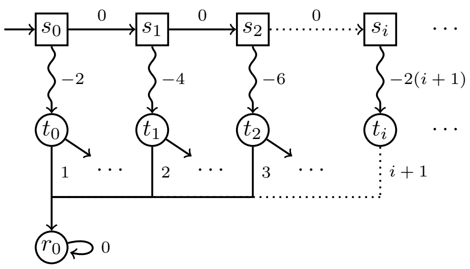
Notice that although finite memory is insufficient for Player 1 in $\mathcal{A}_2$, a step counter allows her to deduce an upper bound on the previous choice of Player 2 and is therefore sufficient. Indeed, since $\mathcal{A}_2$ is finitely branching and every round starts in a unique initial vertex for that round, Player 1 can (over) estimate that all steps of the history so far were spent by her opponent's choice (steps between s_0^i up to some s_j^i and then leading directly to t_0^{i+1}).

In order to construct an arena in which no step-counter strategy is sufficient, we obfuscate possible histories leading to Player 1's choices by making them the same length (see Figure 4).

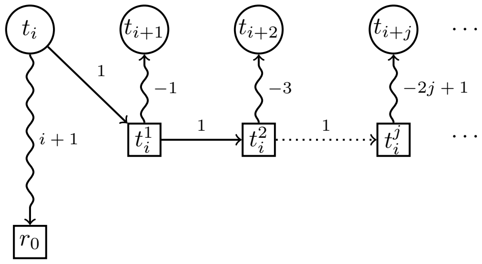
▶ **Lemma 5.** *Consider the arena $\mathcal{A}_3$ depicted in Figure 4. Player 1 has a winning strategy, but no winning step-counter strategy for the objectives* $\underline{\mathsf{TP}}_{>0}$, $\underline{\mathsf{TP}}_{\geq 0}$, $\overline{\mathsf{TP}}_{>0}$, *and* $\overline{\mathsf{TP}}_{\geq 0}$.

Proof. Player 1 only makes relevant choices at vertices t_i, and the choice is whether to delay (move to t_{i+1}) or exit (move to r_0). A winning (finite-memory) strategy for all mentioned objectives is to delay twice and then exit. Indeed, any history leading to t_i has total payoff of at least $-i-1$. By delaying twice and then exiting, Player 1 guarantees that the sink vertex r_0 is reached and the total payoff collected on the way is at least 1.

Conversely, any strategy σ of Player 1 that is based solely on a step counter cannot distinguish histories leading to the same vertex t_i. Let us assume that σ does not choose to



■ **Figure 5** The arena $\mathcal{A}_4$. Arrows $s_i \overset{-2(i+1)}{\rightsquigarrow} t_i$ are shorthand for paths of length $2i+3$ with total payoff $-2(i+1)$. From a vertex t_i, Player 1 either exits to r_0 or moves to the gadget in Figure 6.



■ **Figure 6** The delay gadget from vertex t_i in arena $\mathcal{A}_4$. The arrows from t_i^j to t_{i+j} are shorthand for paths of length $2j$ and payoff $-2j+1$.

avoid r_0 indefinitely, as doing so would result in a negative total payoff, which is losing for her. Then there is at least one vertex t_i from which the strategy exits. Player 2 can exploit this by going there via s_i. The resulting play has a negative total payoff. ◀

We now extend the previous examples to show that even access to both a step counter and finite memory is not sufficient for Player 1. The construction below is stated for the total-payoff objective $\underline{\mathsf{TP}}_{\geq 0}$, and also works for $\underline{\mathsf{TP}}_{>0}$. The main idea is to require Player 1 to delay more than a constant number of times, as dictated by Player 2's initial move.

▶ **Definition 6.** *Let $\mathcal{A}_4$ be the arena from Figure 5. It has a similar high-level structure to $\mathcal{A}_3$ with different weights, and with more complex gadgets (Figure 6) between vertices t_i. At each vertex t_i, Player 1 decides between two actions:*
1. *to exit to r_0 and gain payoff $i+1$ by doing so, or*
2. *to delay to some vertex t_{i+j} where $j > 0$ is chosen by Player 2, and gain payoff $-j+1$.*

Notice that, after Player 2 moved down from vertex s_k, Player 1 can (only) win by delaying at least $k+1$ times (which we show in Lemma 8). We will show that the gadgets allow Player 2 to confuse any strategy of Player 1 that is only based on a step counter and finite memory. Without them, the current vertex t_i together with finite extra memory would allow Player 1 to approximate how many delays she has chosen so far and therefore allow her to win with a finite-memory strategy.[3]

A simple counting argument shows that all paths from s_0 to a vertex t_k have the same length (proof in Appendix A). By Lemma 1, it implies that a step counter is useless in $\mathcal{A}_4$.

▶ **Lemma 7.** *For every t_k in arena $\mathcal{A}_4$, all paths from s_0 to t_k have the same length.*

The following lemma will be used to argue that Player 1 wins, albeit with infinite memory.

▶ **Lemma 8.** *From a vertex t_i, if Player 2 does not stay forever in a gadget, the strategy σ_k of Player 1 that enters the delay gadget exactly $k \in \mathbb{N}$ times achieves a total payoff of exactly $i+k+1$ in r_0.*

[3] The idea would be to partition t_i's into (growing) intervals, so that each interval is picked so large that it is safe to exit from any vertex after the interval if the play entered a vertex before or at the start of that interval. The strategy is then to keep on delaying to t_i's until the first vertex in two intervals have been seen and then exit. This requires 3 memory states.

Proof. Assume that Player 2 never stays forever in a gadget (which would be winning for Player 1 for all quantitative objectives considered). The total payoff on the path from t_i to the next vertex t_{i+j} is $-j+1$. Suppose Player 1 delays k times and let $j(1), j(2), \dots, j(k)$ be the lengths of the intermediate paths through gadgets, as chosen by Player 2. That is, the play ends up in vertex t_{i+l} for $l = \sum_{c=1}^{k} j(c)$ and has gained payoff $\sum_{c=1}^{k}((-j(c)+1)) = -l+k$. After k delays, exiting to r_0 from vertex t_{i+l} gives an immediate payoff of $i+l+1$. The total payoff from t_i to r_0 is thus $(-l+k)+(i+l+1) = i+k+1$. ◀

▶ **Lemma 9.** *Consider the game played on $\mathcal{A}_4$. Then, from vertex s_0,*

1. *Player 1 wins for objective $\underline{\mathsf{TP}}_{\geq 0}$;*
2. *every strategy of Player 1 that is based on (the product of) a step counter and finite memory is losing for $\underline{\mathsf{TP}}_{>-\infty}$.*

Proof. For point (1), let σ be the Player 1 strategy that, upon observing history $s_0 \xrightarrow{*} s_k \to t_k$, switches to the finite-memory strategy σ_{k+1} from the previous lemma (delay $k+1$ times and then exit). Consider any play consistent with this strategy σ. Either Player 2 never moves to a vertex t_k, and then the total payoff is 0, which is winning for Player 1 for $\underline{\mathsf{TP}}_{\geq 0}$. Otherwise, a vertex t_k is reached (and accordingly, the payoff until reaching it is $-2(k+1)$). Using σ_{k+1}, Player 1 guarantees a lim inf total payoff of at least 0 on any continuation: either Player 2 never leaves some gadget and the total payoff is $+\infty$, or Player 1 exits to r_0 after $k+1$ delays, which adds $k+(k+1)+1 = 2(k+1)$ to the total payoff by Lemma 8. In this second case, the total payoff is therefore $-2(k+1)+2(k+1) = 0$.

For point (2), by Lemmas 1 and 7, it suffices to show that every finite-memory strategy of Player 1 is losing. Consider now any such strategy σ_1 of Player 1 with memory of size $K \in \mathbb{N}$ and memory update function δ. We will show that there exists a strategy σ_2 for Player 2 that is winning against σ_1. Player 2's strategy is determined by 1) the initial choice of t_j it visits and 2) which vertex t_{i+j} to select in the gadgets (Figure 6) when Player 1 delays from vertex t_i. We show the existence of suitable choices by employing an argument based on the infinite Ramsey theorem, as follows.

First, δ defines naturally, for any history $h \in E^*$, a function $\delta_h \colon M \to M$ that specifies how the memory is updated when observing this history (formally, $\delta_h(m) = \delta^*(m, h)$). Further, for every $i \geq 0$ there is a function $f_i \colon M \to \{0,1\}$ that describes for which memory states the strategy σ_1 chooses to delay or exit from t_i (formally, $f_i(m)$ equals 1 if $\sigma_1(t_i, m) = (t_i, i+1, r_0)$, and 0 otherwise). As $|M| = K \in \mathbb{N}$, there are only finitely many distinct such functions f_i and δ_h. Consider now the edge-labelled graph G consisting of all vertices $t_i, i \geq 0$, and where for any two $i, j \in \mathbb{N}$, the edge between t_i and t_{i+j} is labelled by the pair (f_i, δ_h) where $h = t_i \to t_i^1 \to \cdots \to t_i^j \to t_{i+j}$ is the history through the delay gadget in $\mathcal{A}_4$.

Recall the infinite Ramsey theorem: If one labels all edges of the complete (undirected and countably infinite) graph with finitely many colours, then there exists an infinite monochromatic subgraph. Applying this to our graph G yields an infinite subgraph, say with vertices $t_{\ell(i)}$ identified by $\ell \colon \mathbb{N} \to \mathbb{N}$, where all edges have the same label. W.l.o.g., assume that $\ell(0) \geq K$ and $\ell(i+1) > \ell(i)+1$ for all $i \geq 0$. Based on this, the strategy σ_2 of Player 2 will 1) initially move to $t_{\ell(0)}$ and 2) whenever Player 1 chooses to delay from $t_{\ell(i)}$ then Player 2 moves to vertex $t_{\ell(i+1)}$. Now consider the play ρ consistent with both strategies σ_1 and σ_2. There are two cases. Either along this play Player 1 chooses to exit from some vertex $t_{\ell(j)}, j < K$, or not. If she exits too early (after delaying only $j < K$ times), then the total payoff after exiting is exactly $-2(\ell(0)+1)+(\ell(0)+j+1) = -\ell(0)+j-1$ by Lemma 8, which is < 0 as $\ell(0) \geq K > j$. Hence, the play is won by Player 2. Alternatively, if along the play, Player 1 delays at least K times then, by the pigeonhole principle, there is at least one memory mode that she revisits. More precisely, the play visits vertices $t_{\ell(i)}$ and $t_{\ell(j)}$, $i < j < K$ in the same memory mode. Recall that the functions $f_{\ell(i)}$ are all identical for $i \geq 0$. It follows that the play will continue visiting vertices $t_{\ell(k)}, k \geq 0$ only and never exit to r_0. Finally, observe that in any delay gadget from a vertex $t_{\ell(i)}$, the path to vertex $t_{\ell(i+1)}$ has total payoff of $1 - (\ell(i+1) - \ell(i))$. Consequently, the infinite play ρ that visits all $t_{\ell(i)}$ will be such that $\underline{\mathsf{TP}}(\rho) = -\infty$ and is losing for Player 1. ◀

▶ **Theorem 10.** *Strategies based on a step counter and finite memory are not sufficient for Player 1 in games with finitely branching arenas and objectives* $\underline{\mathsf{TP}}_{\geq 0}$ *or* $\underline{\mathsf{TP}}_{>0}$*.*

Proof. For $\underline{\mathsf{TP}}_{\geq 0}$ this follows directly from Lemma 9. For $\underline{\mathsf{TP}}_{>0}$, just extend the arena by a new initial vertex s_{-1} with sole outgoing edge $s_{-1} \xrightarrow{1} s_0$ to ensure that the play in which Player 2 never moves to a vertex t_i is won by Player 1. ◀

4 Open objectives

The quantitative objectives defined in Section 2 all belong to the second or third level of the Borel hierarchy, and the strategy complexity of such objectives is not yet well understood. However, they use as building blocks objectives from the first level of the Borel hierarchy (i.e., open and closed objectives), for which there already exist characterisations of memory requirements. We recall some of these results for the memory structures that we study.

Step-monotonicity Let $O \subseteq C^\omega$ be an objective. For two finite words $w_1, w_2 \in C^*$, we write $w_1 \preceq_O w_2$ if for all $w \in C^\omega$, $w_1 w \in O$ implies $w_2 w \in O$ (meaning that the winning continuations of w_1 are included in those of w_2). The relation $\preceq_O$ is a preorder and satisfies that for $w_1, w_2 \in C^*$ and $c \in C$, $w_1 \preceq_O w_2$ implies $w_1 c \preceq_O w_2 c$ (i.e., it is a “congruence”). We write $w_1 \prec_O w_2$ if $w_1 \preceq_O w_2$ but $w_2 \not\preceq_O w_1$. We say that two finite words $w_1, w_2 \in C^*$ are *comparable for* $\preceq_O$ if $w_1 \preceq_O w_2$ or $w_2 \preceq_O w_1$. We extend preorder $\preceq_O$ to histories: we write $h_1 \preceq_O h_2$ if $\mathsf{col}(h_1) \preceq_O \mathsf{col}(h_2)$.

We say that an objective O is *step-monotonic* if for any two finite words $w_1, w_2 \in C^*$ such that $|w_1| = |w_2|$, w_1 and w_2 are comparable for $\preceq_O$. In other words, for any two finite words that are read up to the same state of a step counter, one of the words must include at least the winning continuations of the other word. This is a specialisation of the $\mathcal{M}$*-strong-monotony* property [4] for the step-counter memory structure $\mathcal{M} = \mathcal{S}$.

▶ **Example 11.** Let $C = \{a, b\}$. The open objective $O = aaC^\omega \cup bbC^\omega$ is *not* step-monotonic, as for $w_1 = a$ and $w_2 = b$, we have that $|w_1| = |w_2|$, but w_1 and w_2 are not comparable for $\preceq_O$. Indeed, a^ω (resp. b^ω) is a winning continuation of w_1 but not w_2 (resp. w_2 but not w_1).

Now, let $C = \mathbb{Q}$ and $s \in \mathbb{N}$. The open objective $O_s = \{w \in C^\omega \mid \exists j \geq s, \mathsf{TP}(w_{\leq j}) \geq 0\}$ (containing all infinite words whose total payoff goes over 0 at some point after s steps) is step-monotonic. Indeed, consider two finite words $w_1, w_2 \in C^*$ such that $|w_1| = |w_2|$. If w_2 already satisfies O_s (i.e., $w_2 C^\omega \subseteq O_s$), then necessarily, $w_1 \preceq_{O_s} w_2$. Similarly, if w_1 already satisfies O_s, then $w_2 \preceq_{O_s} w_1$. When neither w_1 nor w_2 already satisfies O_s, they can be compared by their current total payoff: if $\mathsf{TP}(w_1) \leq \mathsf{TP}(w_2)$, then $w_1 \preceq_{O_s} w_2$. ⌟

▶ Remark 12. Variations of objective O_s are used as building blocks to define quantitative objectives (as can be seen in the descriptions in Figure 2), and will be considered again later. An important remark is that $\preceq_{O_s}$ is not completely determined by the current total payoff of words. For instance, if $w_1 = -1, 0$ and $w_2 = 0, -100$, we have $w_1 \prec_{O_1} w_2$ even though $\mathsf{TP}(w_1) > \mathsf{TP}(w_2)$. The reason is that w_2 *already satisfies* O_1 after 1 step, and any continuation is therefore winning, despite the current total payoff being lower. ⌟

Step-counter strategies for open objectives In general, the step-monotonicity property is necessary for the uniform sufficiency of step-counter strategies over finitely branching arenas (this is a specialisation of [4, Lemma 5.2] to the step-counter memory structure $\mathcal{S}$). However, the results of [4] do not yield a characterisation for open objectives in full generality. For the special case of the step-counter memory structure, we can actually show a converse: for open objectives, step-monotonicity implies that step-counter strategies suffice over finitely branching arenas. This is what we show over the next three lemmas (proofs in Appendix B).

First, a handy result about open objectives is that in a *finitely branching* arena, any winning strategy already satisfies the objective within a bounded number of steps.

▶ **Lemma 13.** *Let $O \subseteq C^\omega$ be an open objective, $\mathcal{A}$ be a finitely branching arena, and v_0 be an initial vertex in $\mathcal{A}$. If a strategy σ is winning from v_0 for O, then there is $s \in \mathbb{N}$ such that all histories h of length $\geq s$ consistent with σ already satisfy O, i.e., $\mathsf{col}(h)C^\omega \subseteq O$.*

Second, the following lemma shows that for step-monotonic objectives, step-counter strategies can be "locally not worse" than arbitrary strategies.

▶ **Lemma 14.** *Let $O \subseteq C^\omega$ be a step-monotonic objective. Let $\mathcal{A} = (V, V_1, V_2, E)$ be a finitely branching arena, $v_0 \in V$ be an initial vertex, and σ' be any strategy of Player 1 on $\mathcal{A}$. There is a step-counter strategy σ such that, for every history h from v_0 consistent with σ, there is a history h' from v_0 consistent with σ' such that $|h'| = |h|$, $\mathsf{to}(h') = \mathsf{to}(h)$, and $h' \preceq_O h$.*

The previous two lemmas imply that step-counter strategies suffice to win for open, step-monotonic objectives.

▶ **Corollary 15.** *Let $O \subseteq C^\omega$ be an open, step-monotonic objective. Step-counter strategies suffice for O over finitely branching arenas.*

Proof. Let $\mathcal{A}$ be a finitely branching arena. Let v_0 be a vertex from the winning region and σ' be an arbitrary winning strategy from v. By Lemma 13, using that O is open and $\mathcal{A}$ is finite branching, for all histories h of length $\geq s$ consistent with σ', we have $\mathsf{col}(h)C^\omega \subseteq O$.

As O is step-monotonic, let σ be the step-counter strategy provided by Lemma 14. Every history h of length s from v_0 consistent with σ is at least as good (for $\preceq_O$) as a history h' of length s from v_0 consistent with σ'. As h' only has winning continuations, so does h. Therefore, strategy σ is winning from v_0. ◀

5 Prefix-independent $\mathbf{\Pi}_2^0$ objectives

In this section, we show that step-counter strategies suffice for Player 1 for objectives $\overline{\mathsf{MP}}_{\geq 0}$ and $\overline{\mathsf{TP}}_{=+\infty}$. In fact, we give a sufficient condition for when step-counter strategies suffice for Player 1 in finitely branching games where the objectives are prefix-independent and in $\mathbf{\Pi}_2^0$.

Recall that an objective is in $\mathbf{\Pi}_2^0$ if it can be written as $\bigcap_{m \in \mathbb{N}} O_m$ for some open objectives O_m.

▶ **Theorem 16.** *Let $O = \bigcap_{m \in \mathbb{N}} O_m \subseteq C^\omega$ be a prefix-independent $\mathbf{\Pi}_2^0$ objective such that the objectives O_m are open and step-monotonic. Then, step-counter strategies suffice uniformly for O over finitely branching arenas.*

Proof. Let $\mathcal{A} = (V, V_1, V_2, E)$ be a finitely branching arena, and let $v_0 \in V$ be an initial vertex. Let $W_{\mathcal{A},1} \subseteq V$ be the winning region of $\mathcal{A}$ for O. We assume that v_0 is in the winning region $W_{\mathcal{A},1}$, and build a winning *step-counter* strategy from v_0.

We build a winning step-counter strategy $\sigma\colon V_1 \times \mathbb{N} \to E$ from v_0 by induction on parameter m used in the definition of $O = \bigcap_{m\in\mathbb{N}} O_m$. We consider the product arena $\mathcal{A} \otimes \mathcal{S}$, and fix a strategy for increasingly high step values. The inductive scheme is as follows: for every $m \in \mathbb{N}$, we fix σ on $V_1 \times \{0, \ldots, k_m - 1\}$ for some step bound $k_m \in \mathbb{N}$. We ensure that

- along all histories from v_0 consistent with σ of length at most k_m, the history does not leave $W_{\mathcal{A},1}$ (i.e., for all reachable (v, s), we have $v \in W_{\mathcal{A},1}$), and
- the open objectives $O_{m'}$ for $m' \leq m$ are already satisfied within k_m steps (i.e., any history of length k_m consistent with σ only has winning continuations for $O_{m'}$).

For the base case, we may assume that we start the induction at $m = -1$ with $k_{-1} = 0$ and $O_{-1} = C^\omega$. We indeed have that from $(v_0, 0)$, the winning region is not left within $k_{-1} = 0$ step and that the open objective O_{-1} is already satisfied.

Now, assume that for some $m \geq 0$, the above properties hold, so we have already fixed the moves of σ in $\mathcal{A}\otimes\mathcal{S}$ on $V_1 \times \{0, \ldots, k_m - 1\}$, yielding arena $(\mathcal{A}\otimes\mathcal{S})_m$. We first show that in arena $(\mathcal{A}\otimes\mathcal{S})_m$ the vertex $(v_0, 0)$ still belongs to the winning region. We have assumed by induction that the winning region $W_{\mathcal{A},1}$ is not left within k_m steps. This means that for all (v, k_m) reachable from $(v_0, 0)$ in $(\mathcal{A}\otimes\mathcal{S})_m$, v is in $W_{\mathcal{A},1}$. As O is prefix-independent, no matter the history from $(v_0, 0)$ to (v, k_m), there is still a winning strategy from (v, k_m) (recall that no choice for Player 1 has been fixed beyond step k_m). Hence, no matter how Player 2 plays in the first k_m steps, there is still a way to win for O from $(v_0, 0)$.

We therefore take an (arbitrary) winning strategy σ'_{m+1} of Player 1 from $(v_0, 0)$ in $(\mathcal{A}\otimes\mathcal{S})_m$. As σ'_{m+1} is winning for $O = \bigcap_{m\in\mathbb{N}} O_m$, σ'_{m+1} wins in particular for the open O_{m+1}. Since the arena is finitely branching and O_{m+1} is open, applying Lemma 13, there is $k'_{m+1} \in \mathbb{N}$ such that for all histories h' of length $\geq k'_{m+1}$ consistent with σ'_{m+1}, h' already satisfies O_{m+1} (i.e., $\mathsf{col}(h')C^\omega \subseteq O_{m+1}$). As O_{m+1} is step-monotonic, by Lemma 14, there is a step-counter strategy σ_{m+1} such that for every history h from $(v_0, 0)$ consistent with σ_{m+1}, there is a history h' from $(v_0, 0)$ consistent with σ'_{m+1} such that $|h'| = |h|$, $\mathsf{to}(h') = \mathsf{to}(h)$, and $h' \preceq_{O_{m+1}} h$.

To ensure that we fix at least one extra step of the strategy in the inductive step, let $k_{m+1} = \max\{k'_{m+1}, k_m + 1\}$. We extend the definition of σ to play the same moves as σ_{m+1} on $V_1 \times \{k_m, \ldots, k_{m+1} - 1\}$, which also defines $(\mathcal{A}\otimes\mathcal{S})_{m+1}$. We prove the two items of the inductive scheme.

First, σ still does not leave $W_{\mathcal{A},1}$ up to step k_{m+1}: indeed, for every history consistent with σ_{m+1}, there is a history consistent with σ'_{m+1} reaching the same vertex. As σ'_{m+1} is winning and O is prefix-independent, no such vertex can be outside of the winning region.

Second, strategy σ then guarantees O_{m+1} within k_{m+1} steps: after k_{m+1} steps, every history consistent with σ_{m+1} is at least as good for $\preceq_{O_{m+1}}$ as a history of length k_{m+1} of σ'_{m+1}. But every history h' of length k_{m+1} consistent with σ' is such that $\mathsf{col}(h')C^\omega \subseteq O_{m+1}$, and therefore has only winning continuations.

This concludes the induction argument and shows the existence of a winning step-counter strategy from v_0 as we iterate this process for $m \to \infty$.

We now know that for any vertex from the winning region, there is a winning step-counter strategy. The existence of a *uniformly* winning step-counter strategy can be shown using prefix-independence of O; this part of the proof is standard and is detailed in Appendix C. ◀

This theorem applies to $\overline{\mathsf{MP}}_{\geq 0}$ and $\overline{\mathsf{TP}}_{=+\infty}$ (see Appendix D for a full proof).

▶ **Corollary 17.** *Step-counter strategies suffice uniformly for* $\overline{\mathsf{MP}}_{\geq 0}$ *and* $\overline{\mathsf{TP}}_{=+\infty}$.

To illustrate Theorem 16 further, we apply it to a non-quantitative objective.

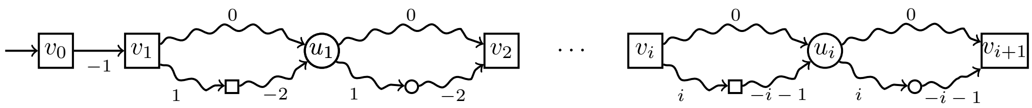


Figure 7 Arena $\mathcal{A}$ used in Example 19. Player 1 has a winning 1-bit strategy for $\overline{\mathsf{TP}}_{\geq 0}$, but no winning step-counter strategy.

▶ **Example 18.** Let C be at most countable. and $O \subseteq C^\omega$ be the objective requiring that all colours are seen infinitely often (it is an intersection of *Büchi conditions*). Formally, $O = \bigcap_{c\in C} \bigcap_{i\geq 1} \bigcup_{j\geq i} \{w = c_1c_2 \ldots \in C^\omega \mid c_j = c\}$. This objective is prefix-independent and in $\mathbf{\Pi}^0_2$: it is the countable intersection of the open, step-monotonic objectives $O_{c,i} = \bigcup_{j\geq i} \{w = c_1c_2 \ldots \in C^\omega \mid c_j = c\}$. By Theorem 16, step-counter strategies suffice over finitely branching arenas for O. This result is relatively tight: finite-memory strategies do not suffice over finitely branching arenas when C is infinite, and step-counter strategies do not suffice over infinitely branching arenas when $|C| = 2$ (see Remark 27, Appendix D for details). ⌟

6 A non-prefix-independent $\mathbf{\Pi}^0_2$ objective

In this section, we consider objective $\overline{\mathsf{TP}}_{\geq 0} = \bigcap_{m\geq 1} \bigcap_{i\geq 1} \bigcup_{j\geq i} \{w \in C^\omega \mid \mathsf{TP}(w_{\leq j}) \geq \frac{-1}{m}\}$ (in $\mathbf{\Pi}^0_2$). Its definition is very close to the one of $\overline{\mathsf{MP}}_{\geq 0}$ from the previous section, but one important difference is that it is not prefix-independent (for instance, $0^\omega \in \overline{\mathsf{TP}}_{\geq 0}$, but $-1, 0^\omega \notin \overline{\mathsf{TP}}_{\geq 0}$). Hence, Theorem 16 does not apply.

As argued in Lemma 5, it turns out that step-counter strategies do not suffice for $\overline{\mathsf{TP}}_{\geq 0}$, even over finitely branching arenas. We show a second example, only suited for this particular objective, illustrating more clearly the trade-off to consider to build simple winning strategies.

▶ **Example 19.** Consider the arena $\mathcal{A}$ in Figure 7. We assume that a play starts in v_0, hence reaching sum of weights -1 in v_1. We assume that a play is decomposed into rounds, where round i corresponds to the choice of Player 2 and Player 1 in v_i and u_i respectively. At each round i, Player 2 and then Player 1 choose either 0, or i followed by $-i-1$. As previously, we can assume that this arena only uses weights in $C = \{-1, 0, 1\}$, and that all histories from v_0 reaching the same vertex have the same length.

Player 1 has a winning strategy, consisting of playing "the opposite" of what Player 2 just played: if Player 2 played the sequence of 0 (resp. $i, -i-1$), then Player 1 replies with $i, -i-1$ (resp. the sequence of 0). This ensures that (i) the current sum of weights in v_i is exactly $-i$ (it starts at -1 in v_1 and decreases by 1 at each round), and (ii) the current sum of weights reaches exactly 0 once during each round, after i is played. This shows that this strategy is winning for $\overline{\mathsf{TP}}_{\geq 0}$. Such a strategy can be implemented with two memory states that simply remember the choice of Player 2 at each round.

As all histories leading to vertices u_i have the same length, a step-counter strategy cannot distinguish the choices of Player 2 (Lemma 1). Any step-counter strategy is losing:

- either Player 1 only plays 0, in which case Player 2 wins by only playing 0, thereby ensuring that the current sum of weights is -1 from v_1 onwards;
- or Player 1 plays $i, -i-1$ at some u_i. In this case, Player 2 wins by only playing $i, -i-1$. This means that the sum of weights decreases by at least 1 at every round, but decreases by 2 in round i. Hence, for $j \geq i$, the sum of weights at round j is at most $-j-1$. Such a sum can never go above 0 again when a player plays $j, -j-1$. ⌟

This example shows that in general, there is a trade-off between "obtaining a high value for a short time, to go above 0 temporarily" and "playing safe in order not to decrease the value too much". Two memory states sufficed: if the opponent just saw a high sum of weights (≥ 0), then we can play it safe temporarily; if the opponent played it safe, we may need to aim for a high value, even if the overall sum decreases. This reasoning generalises to all finitely branching arenas: in general, step-counter + 1-bit strategies suffice for $\overline{\mathsf{TP}}_{\geq 0}$.

▶ **Theorem 20.** *Step-counter + 1-bit strategies suffice for $\overline{\mathsf{TP}}_{\geq 0}$ over finitely branching arenas.*

We provide a proof sketch here (full proof in Appendix E). It follows the same scheme as the proof of Theorem 16, where we inductively fix choices for ever longer histories. However, we need to be more careful not to leave the winning region. As the objective is not prefix-independent, the winning region $W'_{\mathcal{A},1}$ is described not just by a set of vertices, but by pairs of a vertex and current total payoff (i.e., the current sum of weights), i.e, $W'_{\mathcal{A},1} \subseteq V \times \mathbb{Q}$.

We start with a lemma about the sufficiency of memoryless strategies to *stay* in this winning region. Staying in $W'_{\mathcal{A},1}$ is necessary but not sufficient to win for $\overline{\mathsf{TP}}_{\geq 0}$.

▶ **Lemma 21.** *Let $\mathcal{A} = (V, V_1, V_2, E)$ be a finitely branching arena. There exists a memoryless strategy σ_{safe} of Player 1 in $\mathcal{A}$ such that, for every $(v_0, r) \in W'_{\mathcal{A},1}$, σ_{safe} never leaves $W'_{\mathcal{A},1}$ from v_0 with initial weight value r.*

The following lemma is an analogue of Lemma 14, but ensures a stronger property with a more complex memory structure (using an extra bit). It says that locally, with a step-counter + 1-bit strategy, we can guarantee a high value temporarily while staying in the winning region $W'_{\mathcal{A},1}$, generalising the phenomenon of Example 19. The bit is used to aim for a high value (bit value 0) or stay in the winning region (bit value 1) by playing σ_{safe} from Lemma 21.

▶ **Lemma 22.** *Let $\mathcal{A} = (V, V_1, V_2, E)$ be an arena and $v_0 \in V$ be an initial vertex in the winning region of Player 1 for $\overline{\mathsf{TP}}_{\geq 0}$. For all $m \geq 1$, there exists a step-counter + 1-bit strategy σ_m such that σ_m is winning for O_m from v_0 and never leaves $W'_{\mathcal{A},1}$ (i.e., for all histories h from v_0 consistent with σ_m, $(\mathsf{to}(h), \mathsf{TP}(h)) \in W'_{\mathcal{A},1}$).*

The inductive scheme used in the proof of Theorem 20 is similar to that of Theorem 16, building a step-counter + 1-bit strategy $\sigma\colon V_1 \times \mathbb{N} \times \{0, 1\} \to E$.

For $\mathcal{M}$ the product of a step counter and a 1-bit memory structure, consider the product arena $\mathcal{A}' = \mathcal{A} \otimes \mathcal{M}$ (in which the bit updates are not fixed yet, and will be fixed inductively). We have that $(v_0, (0, 0))$ is in the winning region of $\mathcal{A}'$. The inductive scheme is as follows: for infinitely many $m \in \mathbb{N}$, for some step bound $k_m \in \mathbb{N}$, we fix σ on $V_1 \times \{0, \ldots, k_m - 1\} \times \{0, 1\}$, yielding arena $\mathcal{A}'_m$. Using Lemma 22, we ensure that

- along all histories h from v_0 consistent with σ of length at most k_m, $W'_{\mathcal{A},1}$ is not left, and
- the open objective O_m is already satisfied within k_m steps (i.e., any history of length k_m consistent with σ only has winning continuations for O_m).

Iterating this procedure defines a step-counter + 1-bit strategy σ that satisfies O_m for infinitely many $m \geq 1$. As the sequence O_m is decreasing ($O_1 \supseteq O_2 \supseteq \ldots$), we have that σ is winning for O_m for all $m \geq 1$. Hence, σ is winning for $\overline{\mathsf{TP}}_{\geq 0}$.

▶ Remark 23. Unlike for Theorem 16, the upper bound in this section does not apply *uniformly* in general (an arena illustrating this is in Appendix E, Figure 9). ⌟

▶ Remark 24. Over integer weights ($C \subseteq \mathbb{Z}$), $\overline{\mathsf{TP}}_{>0} = \overline{\mathsf{TP}}_{\geq 1} \in \mathbf{\Pi}^0_2$. As $\overline{\mathsf{TP}}_{\geq 1}$ behaves like $\overline{\mathsf{TP}}_{\geq 0}$ (Remark 2), the results from this section apply to $\overline{\mathsf{TP}}_{>0}$ over integer weights. However, for rational weights, $\overline{\mathsf{TP}}_{>0}$ can only be shown to be in $\mathbf{\Sigma}^0_3$, so the above does not apply. ⌟

7 Conclusion

We established whether step-counter strategies (possibly with finite memory) suffice for the objectives $\overline{\mathsf{MP}}_{\geq 0}$, $\underline{\mathsf{TP}}_{>0}$, $\underline{\mathsf{TP}}_{\geq 0}$, $\overline{\mathsf{TP}}_{=+\infty}$, and $\overline{\mathsf{TP}}_{\geq 0}$. To do so, we used the structure of these objectives as sets in the Borel hierarchy, in particular pinpoint the strategy complexity for all classical quantitative objectives on the second level of Borel hierarchy. This leaves open the cases of $\overline{\mathsf{MP}}_{>0}$, $\underline{\mathsf{MP}}_{\geq 0}$, $\overline{\mathsf{TP}}_{>0}$ (over $\mathbb{Q}$), $\underline{\mathsf{TP}}_{=+\infty}$, and $\overline{\mathsf{TP}}_{=+\infty}$, all on the third level. The sufficiency of other less common infinite memory structures, such as *reward counters* [26], could also be investigated.

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A Missing proofs for Section 3

We give formal proofs of Lemmas 3 and 7.

▶ **Lemma 3.** *Over infinitely branching arenas with arbitrary weights, strategies based on a step counter and finite memory are not sufficient for Player 1 for objectives* $\overline{\mathsf{MP}}_{>0}$, $\overline{\mathsf{MP}}_{\geq 0}$, $\overline{\mathsf{TP}}_{=+\infty}$, $\underline{\mathsf{TP}}_{>0}$, $\underline{\mathsf{TP}}_{\geq 0}$, $\overline{\mathsf{TP}}_{>0}$ *and* $\overline{\mathsf{TP}}_{\geq 0}$.

Proof. Let $\mathcal{A}_1$, depicted in Figure 1a, be the arena with three vertices, s controlled by Player 2 and t, q controlled by Player 1, with weights $C = \mathbb{Z}$ and edges $E = (\{s\} \times (-\mathbb{N}) \times \{t\}) \cup (\{t\} \times \mathbb{N} \times \{q\}) \cup \{(q, 0, q)\}$. In this arena, Player 1 has a strategy to win for any total-payoff objective with threshold 0: given the choice of $-i$ by Player 2, Player 1 can respond with $i + 1$, thus winning for $\underline{\mathsf{TP}}_{>0}$, $\underline{\mathsf{TP}}_{\geq 0}$, $\overline{\mathsf{TP}}_{>0}$, and $\overline{\mathsf{TP}}_{\geq 0}$. Notice that, at t, the step-counter value is always 1, providing no useful information. Consider any finite-memory strategy of Player 1. There must be a maximum number k chosen by this strategy at t, against any possible choices of Player 2. Against such a strategy, Player 2 wins by playing $-k - 1$.

We now show the claim for $\overline{\mathsf{MP}}_{>0}$, $\overline{\mathsf{MP}}_{\geq 0}$, and $\overline{\mathsf{TP}}_{=+\infty}$. Consider $\mathcal{A}'_1$ (see Figure 1b), which repeats the process of $\mathcal{A}_1$. Vertices t and s are controlled by Player 1 and Player 2 respectively, with weights $C = \mathbb{Z}$ and edges $E = (\{s\} \times (-\mathbb{N}) \times \{t\}) \cup (\{t\} \times \mathbb{N} \times \{s\})$. That is, the structure enforces strict alternation and each step has an arbitrary finite weight.

Notice that regardless of the players' choices, the number of steps up to round i is always $2i$. Every Player 1 strategy based on a step counter and finite memory thus defines a function $f : \mathbb{N} \to \mathbb{N}$ where $f(i)$ denotes the maximum number chosen in round i. Player 2 can counter such a strategy by always picking $-f(i) - 1$ in round i and thereby ensure a mean payoff $\leq -\frac{1}{2}$ and a total payoff of $-\infty$. ◀

▶ **Lemma 7.** *For every t_k in arena $\mathcal{A}_4$, all paths from s_0 to t_k have the same length.*

Proof. By induction on k. We show that the length of any path from s_0 to t_k is $3(k + 1)$. For $k = 0$, there is only one path of length $3 = 3(k + 1)$. For $k + 1$, the "direct" path $s_0 \xrightarrow{*} s_{k+1} \to t_{k+1}$ evidently has the stated length: $k+1$ steps to reach s_{k+1} then $2(k+1)+3$ steps down to t_{k+1}. Consider any other path $s_0 \xrightarrow{*} t_j \xrightarrow{*} t_{k+1}$ with $j < k + 1$ maximal, i.e., the suffix $t_j \xrightarrow{*} t_{k+1}$ went through the delay gadget from t_j. By induction hypothesis, the prefix up to vertex t_j has length $l_1 = 3(j + 1)$. The suffix from $t_j \to t_j^1 \xrightarrow{*} t_j^{k+1-j} \to t_{k+1}$ has length $l_2 = (k + 1 - j) + 2(k + 1 - j)$. The total length of the path is thus $l_1 + l_2 = 3(j + 1) + 3(k + 1 - j) = 3(k + 2)$ as required. ◀

B Missing proofs for Section 4

In this section, we prove Lemmas 13 and 14.

▶ **Lemma 13.** *Let $O \subseteq C^\omega$ be an open objective, $\mathcal{A}$ be a finitely branching arena, and v_0 be an initial vertex in $\mathcal{A}$. If a strategy σ is winning from v_0 for O, then there is $s \in \mathbb{N}$ such that all histories h of length $\geq s$ consistent with σ already satisfy O, i.e., $\mathsf{col}(h)C^\omega \subseteq O$.*

Proof. Let σ be a strategy winning from v_0 for O. Let T_σ be the set of all histories h from v_0 consistent with σ such that $\mathsf{col}(h)C^\omega \nsubseteq O$. We have that T_σ is a tree, as if $\mathsf{col}(h)C^\omega \subseteq O$, then for h' a longer history with h as a prefix, we also have $\mathsf{col}(h')C^\omega \subseteq O$.

Assume that, for all $s \in \mathbb{N}$, there is h of length $\geq s$ such that $\mathsf{col}(h)C^\omega \nsubseteq O$. Then, T_σ is an infinite tree. By Kőnig's lemma, this tree has an infinite branch. Hence, there is an infinite play ρ consistent with σ such that all finite prefixes h of ρ are such that $\mathsf{col}(h)C^\omega \not\subseteq O$. As O is open, this means that $\mathsf{col}(\rho) \notin O$, so σ is not winning from v_0. ◀

▶ **Lemma 14.** *Let $O \subseteq C^\omega$ be a step-monotonic objective. Let $\mathcal{A} = (V, V_1, V_2, E)$ be a finitely branching arena, $v_0 \in V$ be an initial vertex, and σ' be any strategy of Player 1 on $\mathcal{A}$. There is a step-counter strategy σ such that, for every history h from v_0 consistent with σ, there is a history h' from v_0 consistent with σ' such that $|h'| = |h|$, $\mathsf{to}(h') = \mathsf{to}(h)$, and $h' \preceq_O h$.*

Proof. We define $\sigma \colon V_1 \times \mathbb{N} \to E$ and prove the required property on σ by induction on $\mathbb{N}$. The property is trivially true for histories of length 0 (i.e., just fixing an initial vertex).

We assume that σ has been defined on $V_1 \times \{0, \dots, s-1\}$ for some $s \geq 0$, and that the property holds for histories up to length s. We define σ on histories of length s and prove the properties on histories of length $s+1$ at the same time.

Let h be any history from v_0 consistent with σ of length s. Let $v = \mathsf{to}(h)$. By induction hypothesis, there is a history h' from v_0 consistent with σ' such that $|h'| = s$, $\mathsf{to}(h') = v$, and $h' \preceq_O h$.

If $v \in V_2$, we consider any outgoing edge e of v (which could be a move played by Player 2 after h). The history he has length $s+1$ and is consistent with σ. As h' also ends in $v \in V_2$, the history $h'e$ is also consistent with σ'. As we had $h' \preceq_O h$, we also have $h'e \preceq_O he$ (using that $\preceq_O$ is a congruence). Hence, history $h'e$ satisfies all the required properties: it is consistent with σ', it is of length $s+1$, it ends in $\mathsf{to}(he)$, and it is such that $h'e \preceq_O he$.

If $v \in V_1$, we need to define $\sigma(v, s)$ to see how history h is extended. For $v_1 \in V$, we define a history $h'_{v,s}$ as one of the elements of

$$\min_{\preceq_O}\{h'' \in \mathsf{hists}(\mathcal{A}) \mid h'' \text{ is consistent with } \sigma', \mathsf{from}(h'') = v_0, |h''| = s, \text{ and } \mathsf{to}(h'') = v\}.$$

The set being minimised over is non-empty, as h' is in it. It is also finite, as $\mathcal{A}$ is finitely branching and there are therefore only finitely many histories of fixed length from v_0. Moreover, a minimum exists as all histories of the same length are comparable for $\preceq_O$ due to step-monotonicity. As $\preceq_O$ is a *pre*order, there may be multiple histories that are minimal but equivalent with respect to $\preceq_O$, in which case we can pick any of them.

We define $\sigma(v, s) = \sigma'(h'_{v,s})$. Let he be the one-move continuation of h, where $e = \sigma(v, s)$. We have that $h'_{v,s}e$ is consistent with σ', and satisfies $|h'_{v,s}e| = |he|$ and $\mathsf{to}(h'_{v,s}e) = \mathsf{to}(he)$. Moreover, we had $h' \preceq_O h$ (by the induction hypothesis) and we have $h'_{v,s} \preceq_O h'$ by the choice of minimum. Therefore, $h'_{v,s} \preceq_O h$, so $h'_{v,s}e \preceq_O he$, which ends the proof. ◀

C Uniformly winning strategies

In this section, we prove that winning step-counter strategies can be "uniformised" for prefix-independent objectives. This result easily follows from the following known result [10, Lemma 5] on the uniformisation of *memoryless* strategies for prefix-independent objectives. We rephrase this lemma with our notations. This result was originally stated for both players at the same time, but its proof applies to one player at a time.

▶ **Lemma 25** ([10, Lemma 5])**.** *Let O be a prefix-independent objective, and $\mathcal{A} = (V, V_1, V_2, E)$ be an arena. If, for all $v \in W_{\mathcal{A},1}$ in the winning region of Player 1, there is a winning memoryless strategy from v, then Player 1 has a uniformly winning memoryless strategy in $\mathcal{A}$.*

We obtain a similar result for step-counter strategies by reducing to the memoryless case through the construction of the product arena $\mathcal{A} \otimes \mathcal{S}$.

▶ **Lemma 26.** *Let O be a prefix-independent objective and $\mathcal{A} = (V, V_1, V_2, E)$ be an arena in which, from every $v \in W_{\mathcal{A},1}$ of the winning region of Player 1, Player 1 has a winning step-counter strategy. Then Player 1 has a uniformly winning step-counter strategy.*

Proof. Consider the product arena $\mathcal{A} \otimes \mathcal{S}$. The fact that, from every vertex $v \in W_{\mathcal{A},1}$, Player 1 has a winning step-counter strategy is equivalent to the fact that, from every $(v, 0)$ in $\mathcal{A} \otimes \mathcal{S}$ for $v \in W_{\mathcal{A},1}$, Player 1 has a winning memoryless strategy. The proof of this equivalence is standard; full details can be found in [5, Lemma 2.4].

As O is prefix-independent, we can apply Lemma 25 to arena $\mathcal{A}\otimes\mathcal{S}$ and find a memoryless strategy σ of Player 1 that wins from all vertices in the winning region of $\mathcal{A}\otimes\mathcal{S}$. In particular, it wins from all vertices $(v, 0)$ with $v \in W_{\mathcal{A},1}$. Going back to $\mathcal{A}$, we find that there is a step-counter strategy that wins from all $v \in W_{\mathcal{A},1}$. ◀

D Missing details for Section 5

We prove Corollary 17.

▶ **Corollary 17.** *Step-counter strategies suffice uniformly for $\overline{\mathsf{MP}}_{\geq 0}$ and $\overline{\mathsf{TP}}_{=+\infty}$.*

Proof. Let $C = \mathbb{Q}$. Recall that both $\overline{\mathsf{MP}}_{\geq 0}$ and $\overline{\mathsf{TP}}_{=+\infty}$ are prefix-independent.

We first focus on $\overline{\mathsf{MP}}_{\geq 0}$. A natural way to write $\overline{\mathsf{MP}}_{\geq 0}$ as a $\mathbf{\Pi}^0_2$ objective, starting from its definition, is as the set of infinite words that have infinitely many finite prefixes with mean payoff above $\frac{-1}{m}$ for all $m \geq 1$. Formally,

$$\overline{\mathsf{MP}}_{\geq 0} = \bigcap_{m\geq 1} \bigcap_{i\geq 1} \bigcup_{j\geq i} \left\{ w \in C^\omega \mid \mathsf{MP}(w_{\leq j}) \geq \frac{-1}{m} \right\}.$$

For $m, i \geq 1$, let $O_{m,i} = \bigcup_{j\geq i}\{w \mid \mathsf{MP}(w_{\leq j}) \geq \frac{-1}{m}\}$. Such sets are open, as whether a word belongs to it is witnessed by a finite prefix.

Observe that the objectives $O_{m,i}$ are also step-monotonic. Indeed, if we consider two finite words of the same length, either one has already satisfied $O_{m,i}$, or the one with the greatest current mean payoff has more winning continuations than the other (a similar reasoning was used in Example 11).

Therefore, $\overline{\mathsf{MP}}_{\geq 0}$ is the countable intersection of open, step-monotonic objectives. By Theorem 16, we conclude that a step counter suffices for $\overline{\mathsf{MP}}_{\geq 0}$ in finitely branching arenas.

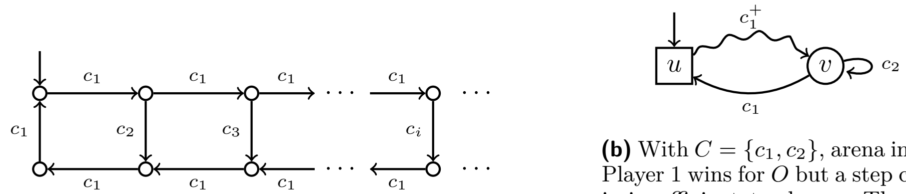
The claim for $\overline{\mathsf{TP}}_{=+\infty}$ can be shown analogously: observe that

$$\overline{\mathsf{TP}}_{=+\infty} = \bigcap_{m\geq 1} \bigcap_{i\geq 1} \bigcup_{j\geq i} \{w \in C^\omega \mid \mathsf{TP}(w_{\leq j}) \geq m\}$$

is in $\mathbf{\Pi}^0_2$ and fix $O_{m,i} = \bigcup_{j\geq i} \{w \in C^\omega \mid \mathsf{TP}(w_{\leq j}) \geq m\}$. ◀

▶ Remark 27 (Further details on Example 18). The $O_{c,i}$ are open because whether an infinite word is in such an objective is witnessed by a finite prefix of the word. Also, the objectives $O_{c,i}$ are step-monotonic: if we take two finite words of the same length, either one already satisfies the objective (in which case, this word is larger for $\preceq_{O_{c,i}}$), or none of the words has already satisfied the objective, in which case both words have exactly the same winning continuations.

We show that finite-memory strategies do not suffice over finitely branching arenas when C is infinite: consider the arena in Figure 8a. In this arena, Player 1 can see all colours infinitely often, but any finite-memory strategy either only stays on the top horizontal line, or never goes beyond some c_i.



(a) With $C = \{c_1, c_2, \ldots\}$, arena in which Player 1 wins for O but finite memory is insufficient to do so.

(b) With $C = \{c_1, c_2\}$, arena in which Player 1 wins for O but a step counter is insufficient to do so. The arrow with label c_1^* indicates that Player 2 can choose any word in language c_1^+.

Figure 8 Arenas illustrating that memory is insufficient for objective O in Remark 27.

We show that step-counter strategies do not suffice over infinitely branching arenas when $|C| = 2$: consider the arena in Figure 8b. Clearly, Player 1 wins by playing c_2 followed by c_1 every time the play reaches v. However, using only a step-counter strategy σ, there needs to be infinitely many s such that $\sigma(v, s) = (v, c_2, v)$ and infinitely many s such that $\sigma(v, s) = (v, c_1, u)$. For any given step-counter strategy, Player 2 can therefore have a winning counterstrategy by ensuring that the play reaches v only in those steps where Player 1 plays (v, c_1, u) immediately. ⌟

E Missing proofs for Section 6

This section is dedicated to the proof of Theorem 20.

▶ **Theorem 20.** *Step-counter + 1-bit strategies suffice for* $\overline{\mathsf{TP}}_{\geq 0}$ *over finitely branching arenas.*

For an arena $\mathcal{A} = (V, V_1, V_2, E)$, let $W'_{\mathcal{A},1} \subseteq V \times \mathbb{Q}$ be the set of *pairs* (v, r) such that there is a winning strategy from v for $\overline{\mathsf{TP}}_{\geq 0}$, assuming the current sum of weights is r. Formally, $(v, r) \in W'_{\mathcal{A},1}$ if there is a strategy σ such that for all plays ρ consistent with σ, $r + \limsup_j(\mathsf{TP}(\mathsf{col}(\rho)_{\leq j})) \geq 0$. For example, in Example 19 (Figure 7), we have $(v_i, -i) \in W'_{\mathcal{A},1}$, but $(v_i, -i-1) \notin W'_{\mathcal{A},1}$. We say that a strategy *never leaves* $W'_{\mathcal{A},1}$ *from* v_0 *with initial weight value* r if for all histories h from v_0 consistent with the strategy, $(\mathsf{to}(h), r + \mathsf{TP}(h)) \in W'_{\mathcal{A},1}$.

We start by proving Lemma 21 about the sufficiency of memoryless strategies to *stay* in this winning region. Note that staying in $W'_{\mathcal{A},1}$ is necessary but not sufficient for a strategy to be winning for $\overline{\mathsf{TP}}_{\geq 0}$.

▶ **Lemma 21.** *Let $\mathcal{A} = (V, V_1, V_2, E)$ be a finitely branching arena. There exists a memoryless strategy σ_{safe} of Player 1 in $\mathcal{A}$ such that, for every $(v_0, r) \in W'_{\mathcal{A},1}$, σ_{safe} never leaves $W'_{\mathcal{A},1}$ from v_0 with initial weight value r.*

Proof. We define a memoryless strategy $\sigma_{\mathsf{safe}} : V_1 \to E$ with the required properties. Let $v \in V_1$. Assume that the set $R_v = \{r \in \mathbb{Q} \mid (v, r) \in W'_{\mathcal{A},1}\}$ is not empty (if it is, we define $\sigma(v)$ arbitrarily). Notice that R_v is upwards closed (i.e., if $r \in R_v$ and $r' \geq r$, then $r' \in R_v$). Let $r_v = \inf R_v$. If $r_v \in R_v$, there exists a strategy σ^v that wins for $\overline{\mathsf{TP}}_{\geq 0}$ from v with current weight value r_v; we fix $\sigma_{\mathsf{safe}}(v) = \sigma^v(\lambda_v)$ (we recall that λ_v is the empty history from v). If $r_v \notin R_v$, for every $n \geq 1$, as $(v, r_v + \frac{1}{n}) \in W'_{\mathcal{A},1}$, there exists a strategy σ_n^v that wins for $\overline{\mathsf{TP}}_{\geq 0}$ from v with current weight value $r_v + \frac{1}{n}$. Consider the infinite sequence of edges $\sigma_1^v(\lambda_v), \sigma_2^v(\lambda_v), \ldots$; as $\mathcal{A}$ is finitely branching, one of the edges outgoing from v appears infinitely often along this sequence. We define $\sigma_{\mathsf{safe}}(v)$ to be such an edge.

We now show that σ_{safe} satisfies the property from the statement. Let $(v_0, r) \in W'_{\mathcal{A},1}$. Let h be a history from v_0 consistent with σ_{safe}. We show by induction on the length of h that $r + \mathsf{TP}(h) \in R_{\mathsf{to}(h)}$. For the base case, we have that $r \in R_{v_0}$.

Assume now that he is a history from v_0 consistent with σ_{safe} of length $|h| + 1$ and that $r + \mathsf{TP}(h) \in R_{\mathsf{to}(h)}$. We have that $\inf R_{\mathsf{to}(h)} \leq r + \mathsf{TP}(h)$. If $\inf R_{\mathsf{to}(h)} \in R_{\mathsf{to}(h)}$, then $\inf R_{\mathsf{to}(h)} + \mathsf{col}(e) \in R_{\mathsf{to}(e)}$ by definition of $\sigma_{\mathsf{safe}}(\mathsf{to}(h))$, so $r + \mathsf{TP}(he) \in R_{\mathsf{to}(e)}$ (as $R_{\mathsf{to}(e)}$ is upwards closed). If $\inf R_{\mathsf{to}(h)} \notin R_{\mathsf{to}(h)}$, then $\inf R_{\mathsf{to}(h)} < r + \mathsf{TP}(h)$. So we can find $n \in \mathbb{N}$ such that $\inf R_{\mathsf{to}(h)} + \frac{1}{n} \leq r + \mathsf{TP}(h)$ and such that $\sigma_{\mathsf{safe}}(\mathsf{to}(h)) = \sigma_n^{\mathsf{to}(h)}(h_{\mathsf{to}(h)})$. With a similar observation as the previous case, we obtain that $r + \mathsf{TP}(he) \in R_{\mathsf{to}(e)}$. ◀

Lemma 22 is an analogue of Lemma 14, but ensures a stronger property with a more complex memory structure (using an extra bit). It says that locally, with a step-counter + 1-bit strategy, we can guarantee a high value temporarily while staying in the winning region $W'_{\mathcal{A},1}$. It generalises the phenomenon of Example 19, in which we observed that 1 bit is exactly what we need to either stay in the winning region or aim for a high value. We will later use it in the induction step in the proof of Theorem 20.

We first write $\overline{\mathsf{TP}}_{\geq 0}$ in a different way: observe that

$$\overline{\mathsf{TP}}_{\geq 0} = \bigcap_{m \geq 1} \bigcup_{j \geq m} \left\{ w \in C^\omega \mid \mathsf{TP}(w_{\leq j}) \geq \frac{-1}{m} \right\}, \tag{1}$$

where the variable m is used both for the $-\frac{1}{m}$ lower bound and for the m lower bound on the step count. Indeed, this also enforces that, for arbitrarily long prefixes, the current total payoff goes above values arbitrarily close to 0. For $m \geq 1$, let $O_m = \bigcup_{j \geq m} \{w \mid \mathsf{TP}(w_{\leq j}) \geq \frac{-1}{m}\}$ be the open set used in the definition of $\overline{\mathsf{TP}}_{\geq 0}$ in (1). To better understand $\preceq_{O_m}$, recall that as in Remark 12, for any two words $w_1, w_2 \in C^*$ with $|w_1| = |w_2|$, we have $w_1 \preceq_{O_m} w_2$ if and only if

$$w_2 C^\omega \subseteq O_m \quad \text{or} \quad (w_1 C^\omega \not\subseteq O_m \text{ and } \mathsf{TP}(w_1) \leq \mathsf{TP}(w_2)). \tag{2}$$

The first term of the disjunction corresponds to the case where w_2 has a prefix that already satisfies O_m, and hence every continuation is winning for O_m. The second term then covers the case where none of the two words has satisfied O_m so far and w_2 has a current sum of weights at least as high as w_1.

► **Lemma 22.** *Let $\mathcal{A} = (V, V_1, V_2, E)$ be an arena and $v_0 \in V$ be an initial vertex in the winning region of Player 1 for $\overline{\mathsf{TP}}_{\geq 0}$. For all $m \geq 1$, there exists a step-counter + 1-bit strategy σ_m such that σ_m is winning for O_m from v_0 and never leaves $W'_{\mathcal{A},1}$ (i.e., for all histories h from v_0 consistent with σ_m, $(\mathsf{to}(h), \mathsf{TP}(h)) \in W'_{\mathcal{A},1}$).*

Proof. Let σ' be an arbitrary winning strategy from v_0 (which exists as v_0 is in the winning region, i.e., $(v_0, 0) \in W'_{\mathcal{A},1}$). Let σ_{safe} be a memoryless strategy on $\mathcal{A}$ staying in $W'_{\mathcal{A},1}$ given by Lemma 21. We consider the "step-counter + 1-bit" memory structure $\mathcal{M} = (\mathbb{N} \times \{0, 1\}, (0, 0), \delta)$ (we have not yet defined how and when δ updates the bit, which will come later in the proof).

We define the strategy σ_m based on $\mathcal{M}$ inductively from v_0, on the set $V_1 \times \mathbb{N} \times \{0, 1\}$. As defined above, the initial memory bit is 0. The goal is to guarantee the following two properties: for all histories h consistent with σ_m from v_0,

1. we have $(\mathsf{to}(h), \mathsf{TP}(h)) \in W'_{\mathcal{A},1}$, that is, the strategy does not leave the winning region for $\overline{\mathsf{TP}}_{\geq 0}$;

2. if the memory bit after h is 0, there exists a history h' consistent with σ' from v_0 such that $|h'| = |h|$, $\mathsf{to}(h') = \mathsf{to}(h)$, and $h' \preceq_{O_m} h$.

The properties clearly hold on the empty history λ_{v_0} from v_0 (for which the bit value is still 0). We will prove the two properties inductively after defining the strategy σ_m.[4]

Assume that σ_m has already been defined on $V_1 \times \{0, \ldots, s-1\} \times \{0, 1\}$. Let h be a history from v_0 consistent with σ_m of length s. We write v for $\mathsf{to}(h)$ for brevity. To extend σ_m, we need to define $\sigma_m(v, s, 0)$ and $\sigma_m(v, s, 1)$ for $v \in V_1$, and we need to define when the memory bit is updated. If the memory bit after h is 0, we define a history

$$h'_{v,s} \in \min_{\preceq_{O_m}} \{h' \mid h' \text{ is consistent with } \sigma', \mathsf{from}(h') = v_0, |h'| = s, \text{ and } \mathsf{to}(h') = v\}.$$

This set is non-empty due to the induction hypothesis 2., and well-ordered due to step-monotonicity of O_m, so $h'_{v,s}$ is well-defined. If $v \in V_1$, we define $\sigma_m(v, s, 0) = \sigma'(h'_{v,s})$ and $\sigma_m(v, s, 1) = \sigma_{\mathsf{safe}}(v)$.

We now define the bit update. Let e be a possible edge taken from v after history h (either e is the edge taken by σ_m after h if $v \in V_1$, or e is any possible outgoing edge of v if $v \in V_2$). When the bit is 1, we never change the bit (i.e., $\delta((s, 1), e) = (s + 1, 1)$). Assume now that the bit of σ_m is 0 after h. We update the bit from 0 to 1 (i.e., define $\delta((s, 0), e) = (s + 1, 1)$) if and only if $\mathsf{col}(h'_{v,s}e)C^\omega \subseteq O_m$ (intuitively, this guarantees that O_m is now satisfied even following σ').

Now that we have fully defined the next step of σ_m after h, it is left to prove the two properties. To do so, we first prove the following property:

$$\text{if the memory bit after } h \text{ is 0, then } \mathsf{TP}(h'_{v,s}) \leq \mathsf{TP}(h). \tag{3}$$

We prove the contrapositive. Assume that $\mathsf{TP}(h) < \mathsf{TP}(h'_{v,s})$. Observe that this cannot happen if $|h| = 0$. Let $\widetilde{h}e$ be the shortest prefix of h such that $\mathsf{TP}(h'_{\mathsf{to}(\widetilde{h}),|\widetilde{h}|}) \leq \mathsf{TP}(\widetilde{h})$ but $\mathsf{TP}(\widetilde{h}e) < \mathsf{TP}(h'_{\mathsf{to}(\widetilde{h}e),|\widetilde{h}e|})$. We have that $h'_{\mathsf{to}(\widetilde{h}),|\widetilde{h}|}e$ is consistent with σ' and $\mathsf{TP}(h'_{\mathsf{to}(\widetilde{h}),|\widetilde{h}|}e) \leq \mathsf{TP}(\widetilde{h}e)$. Combining the last two inequalities, we obtain that $\mathsf{TP}(h'_{\mathsf{to}(\widetilde{h}),|\widetilde{h}|}e) < \mathsf{TP}(h'_{\mathsf{to}(\widetilde{h}e),|\widetilde{h}e|})$. However, by definition of $h'_{\mathsf{to}(\widetilde{h}e),|\widetilde{h}e|}$, we have that $h'_{\mathsf{to}(\widetilde{h}e),|\widetilde{h}e|} \preceq_{O_m} h'_{\mathsf{to}(\widetilde{h}),|\widetilde{h}|}e$.

Using the discussion about $\preceq_{O_m}$ from (2), this necessarily implies that $\mathsf{col}(h'_{\mathsf{to}(\widetilde{h}),|\widetilde{h}|}e)C^\omega \subseteq O_m$. So the bit was set to 1 after $\widetilde{h}e$, which means that the bit is still 1 after h.

We now prove the two inductive properties.

1. By induction hypothesis, we have $(\mathsf{to}(h), \mathsf{TP}(h)) \in W'_{\mathcal{A},1}$.
   If $v \in V_2$, then for any possible edge e from v, we still have $(\mathsf{to}(he), \mathsf{TP}(he)) \in W'_{\mathcal{A},1}$ by definition of $W'_{\mathcal{A},1}$. We now assume $v \in V_1$ and distinguish whether the bit value after h is 0 or 1.
   If the bit is 0 after h, then by (3), $\mathsf{TP}(h'_{v,s}) \leq \mathsf{TP}(h)$. Let $e = \sigma_m(v, s, 0) = \sigma'(h'_{v,s})$; we have $\mathsf{TP}(h'_{v,s}e) \leq \mathsf{TP}(he)$. As σ' is a winning strategy for $\overline{\mathsf{TP}}_{\geq 0}$, necessarily, $(\mathsf{to}(h'_{v,s}e), \mathsf{TP}(h'_{v,s}e)) \in W'_{\mathcal{A},1}$. As $\mathsf{to}(h'_{v,s}e) = \mathsf{to}(he)$ and $\mathsf{TP}(h'_{v,s}e) = \mathsf{TP}(h'_{v,s}) + \mathsf{col}(e) \leq \mathsf{TP}(h) + \mathsf{col}(e) = \mathsf{TP}(he)$, we have $(\mathsf{to}(he), \mathsf{TP}(he)) \in W'_{\mathcal{A},1}$.

[4] As a side note, we briefly comment on how these two properties can be interpreted on the game of Example 19 (Figure 7). Observe that following the 0 edges guarantees 1. for Player 1 (it ensures a higher sum of weights), but does not guarantee 2. (as h may not have already satisfied one of the O_m, unlike all histories h'). On the other hand, the first time Player 1 plays $i, -i-1$, it ensures 2. (some O_m is already satisfied during this round), but not 1. (it may leave the winning region $W'_{\mathcal{A},1}$ if Player 2 also played $i, -i-1$). This is why the bit is necessary to guarantee both 1. and 2.

If the bit is 1 after history h, then σ_m imitates σ_{safe}, so $(\mathsf{to}(he), \mathsf{TP}(he)) \in W'_{\mathcal{A},1}$ by definition of σ_{safe}.

2. Let e be a possible edge after h. Assume the memory bit after he is 0. This implies that the bit was also 0 after h. By induction hypothesis, there exists a history h' consistent with σ' from v_0 such that $|h'| = |h|$, $\mathsf{to}(h') = \mathsf{to}(h)$, and $h' \preceq_{O_m} h$. We show that $h'_{v,s}e$ satisfies the desired property for he; we already have by definition that $|h'_{v,s}e| = |he|$ and $\mathsf{to}(h'_{v,s}e) = \mathsf{to}(he)$. Moreover, as the bit is 0, $e = \sigma_m(v, s, 0) = \sigma'(h'_{v,s})$, so $h'_{v,s}e$ is also consistent with σ'. By definition of $h'_{v,s}$, we also have $h'_{v,s} \preceq_{O_m} h'$, therefore $h'_{v,s} \preceq_{O_m} h$. Hence, $h'_{v,s}e \preceq_{O_m} he$.

This shows the two inductive properties above. We still have to show that σ_m is winning for O_m from v_0 and never leaves $W'_{\mathcal{A},1}$.

We prove that σ_m wins for O_m. Observe that σ' wins in particular for O_m (as σ' wins for $\overline{\mathsf{TP}}_{\geq 0}$). As O_m is open and $\mathcal{A}$ is finitely branching, this means that all histories consistent with σ' already satisfy O_m after a bounded number of steps (Lemma 13). This means that for s sufficiently large, all histories $h'_{v,s}$ used to define σ_m are such that $\mathsf{col}(h'_{v,s})C^\omega \subseteq O_m$. In particular, for s sufficiently large, due to how the bit update from 0 to 1 is defined, all histories h consistent with σ_m reach a bit value of 1. We show that any history h consistent with σ_m with bit value 1 is such that $\mathsf{col}(h)C^\omega \subseteq O_m$. Let h be such a history, and let $\widetilde{h}$ be its longest prefix with bit value still at 0. If $\mathsf{col}(\widetilde{h})C^\omega \subseteq O_m$, we are done. If $\mathsf{col}(\widetilde{h})C^\omega \not\subseteq O_m$, as the bit value is still 0, there is $\widetilde{h}'$ consistent with σ' such that $|\widetilde{h}'| = |\widetilde{h}|$, $\mathsf{to}(\widetilde{h}') = \mathsf{to}(\widetilde{h})$, and $\widetilde{h}' \preceq_{O_m} \widetilde{h}$. We have that $h'_{\mathsf{to}(\widetilde{h}),|\widetilde{h}|} \preceq_{O_m} \widetilde{h}'$, so $h'_{\mathsf{to}(\widetilde{h}),|\widetilde{h}|} \preceq_{O_m} \widetilde{h}$. In particular, $\mathsf{col}(h'_{\mathsf{to}(\widetilde{h}),|\widetilde{h}|})C^\omega \not\subseteq O_m$ as well. By (2), this implies that $\mathsf{TP}(h'_{\mathsf{to}(\widetilde{h}),|\widetilde{h}|}) \leq \mathsf{TP}(\widetilde{h})$. Let e be the next edge in h after $\widetilde{h}$. As the bit after $\widetilde{h}e$ is 1, this means that $\mathsf{col}(h'_{\mathsf{to}(\widetilde{h}),|\widetilde{h}|}e)C^\omega \subseteq O_m$. As additionally, $|\widetilde{h}e| = |h'_{\mathsf{to}(\widetilde{h}),|\widetilde{h}|}e|$ and $\mathsf{TP}(\widetilde{h}e) \geq \mathsf{TP}(h'_{\mathsf{to}(\widetilde{h}),|\widetilde{h}|}e) \geq -\frac{1}{m}$, we have that $\mathsf{col}(\widetilde{h}e)C^\omega \subseteq O_m$. As h is a continuation of $\widetilde{h}e$, we also have that $\mathsf{col}(h)C^\omega \subseteq O_m$.

We have shown that all histories consistent with σ_m reach bit value 1 within a bounded number of steps, and that bit value 1 indicates that any continuation is winning O_m. This shows that σ_m is winning for O_m.

The fact that σ_m never leaves $W'_{\mathcal{A},1}$ from v_0 is a direct consequence of 1. ◀

Proof of Theorem 20. Using the previous lemma, we now prove Theorem 20. Let $\mathcal{A} = (V, V_1, V_2, E)$ be an arena and $v_0 \in V$ be an initial vertex in the winning region of Player 1 for $\overline{\mathsf{TP}}_{\geq 0}$. Let the $\mathcal{M} = (\mathbb{N} \times \{0,1\}, (0,0), \delta)$ be the "step-counter + 1-bit" memory structure (we still need to define how and when δ updates the bit). We show that there is a winning step-counter + 1-bit strategy σ (i.e., σ is based on $\mathcal{M}$) from v_0.

We build a winning step-counter + 1-bit strategy $\sigma\colon V_1 \times \mathbb{N} \times \{0,1\} \to E$ inductively. Consider the product arena $\mathcal{A}' = \mathcal{A} \otimes \mathcal{M}$ (in which the bit updates are not fixed yet, and will be fixed inductively). We have that $(v_0, (0,0))$ is in the winning region of $\mathcal{A}'$. The inductive scheme is as follows: for infinitely many $m \in \mathbb{N}$, for some step bound $k_m \in \mathbb{N}$, we fix σ on $V \times \{0, \ldots, k_m - 1\} \times \{0,1\}$, yielding arena $\mathcal{A}'_m$. We ensure that

- along all histories h from v_0 consistent with σ of length at most k_m, $W'_{\mathcal{A},1}$ is not left (i.e., $(\mathsf{to}(h), \mathsf{TP}(h)) \in W'_{\mathcal{A},1}$), and
- the open objective O_m is already satisfied within k_m steps (i.e., any history of length k_m consistent with σ only has winning continuations for O_m).

For the base case, we may assume that we start the induction at $m = -1$ with $k_{-1} = 0$ and $O_{-1} = C^\omega$. We indeed have that from $(v_0, (0,0))$, the winning region $W'_{\mathcal{A},1}$ is not left within $k_{-1} = 0$ step and that the open objective O_{-1} is already satisfied.

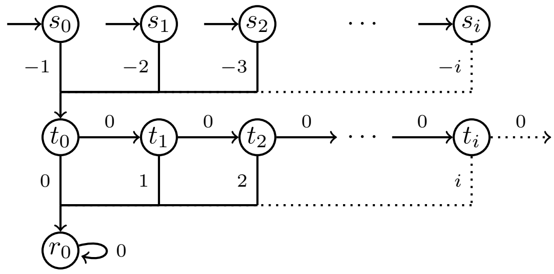


Figure 9 Arena in which Player 1 has no *uniformly* winning step-counter + 1-bit strategy.

Assume that we have fixed σ in $\mathcal{A}'$ (decisions and bit updates) up to some step bound k_m, that $W'_{\mathcal{A},1}$ is not left within k_m steps, and that O_m is already satisfied for all histories of length k_m. We reset the bit to 0 after exactly k_m steps: for $v \in V$ and $b \in \{0,1\}$, we define $\delta(v,(k_m-1,b)) = (k_m, 0)$. Fixing σ up to bound k_m defines an arena $\mathcal{A}'_m$.

As $W'_{\mathcal{A},1}$ is not left within the first k_m steps, and as no decisions have been fixed after k_m steps, vertex $(v_0,(0,0))$ is still in the winning region for $\overline{\mathsf{TP}}_{\geq 0}$. Let $m' = k_m + 1$. We apply Lemma 22: there exists a step-counter + 1-bit strategy $\sigma_{m'}$ such that $\sigma_{m'}$ is winning for $O_{m'}$ from $(v_0,(0,0))$ and never leaves $W'_{\mathcal{A},1}$. Observe that there are no decisions to make in $\mathcal{A}'_m$ before k_m steps have elapsed, and that the bit is set to 0 after exactly k_m steps. By Lemma 13, as $\mathcal{A}'_m$ is finitely branching and $O_{m'}$ is open, there is a bound $k_{m'}$ such that any history consistent with $\sigma_{m'}$ of length $k_{m'}$ already satisfies $O_{m'}$. As $O_{m'}$ cannot be already satisfied before step m', we have $k_{m'} \geq k_m + 1$. We define $\mathcal{A}'_{m'}$ by fixing strategy $\sigma'_{m'}$ up to step $k_{m'}$.

Iterating this procedure defines a step-counter + 1-bit strategy σ such that, for infinitely many $m \geq 1$, σ is winning for O_m. As the sequence O_m is decreasing ($O_1 \supseteq O_2 \supseteq \ldots$), we have that σ is winning for O_m for all $m \geq 1$. Hence, σ is winning for $\overline{\mathsf{TP}}_{\geq 0}$. ◀

In general, this proof does not apply uniformly: in the arena of Figure 9, Player 1 has no uniformly winning strategy based on a step counter and finite memory from all s_i simultaneously, as Player 1 needs to exit to r_0 arbitrarily far to the right.