Reachability in Two-Dimensional Unary Vector Addition Systems with States is NL-Complete *

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Abstract

Blondin et al. showed at LICS 2015 that two-dimensional vector addition systems with states have reachability witnesses of length exponential in the number of states and polynomial in the norm of vectors. The resulting guess-and-verify algorithm is optimal (PSPACE), but only if the input vectors are given in binary. We answer positively the main question left open by their work, namely establish that reachability witnesses of pseudo-polynomial length always exist. Hence, when the input vectors are given in unary, the improved guess-and-verify algorithm requires only logarithmic space.

1. Introduction

To quote from Bojańczyk's preface to Schmitz's very recent survey [8], the reachability problem for vector addition systems with states (VASS) 'is one of the most celebrated decidable problems in theoretical computer science'. The interest, though, is not only theoretical: Schmitz has devoted a long section to 'only a small sample of the problems interreducible' with the reachability problem, and the domains of the problems he identifies range over formal languages, logic, concurrent systems and process calculi.

For informative introductions to the fascinating history of the VASS reachability problem, that stretches from the 1970s, we refer the reader to Schmitz [8] and Blondin et al. [1]. In a nutshell, the state of the art when it comes to the problem's complexity hinges on two recent and one old discovery:

- Remarkably, Lipton's EXPSPACE lower bound [6] is still unbeaten.
- The best known upper bound, by Leroux and Schmitz [5], is *cubic Ackermann*, a non-primitive recursive complexity class.
- The largest fixed dimension for which an interesting upper bound is known is 2: Blondin et al. [1] have established that the 2-VASS reachability problem is in PSPACE.

Our contribution is to resolve the main open question that arises from the latter work, and is highlighted by Schmitz [8]. Namely, the headline result of Blondin et al. is that 2-VASS reachability is PSPACE-complete, but that is provided the input to the problem is succinct, i.e. the integers that specify the action, source and target vectors are given in binary. When the encoding is unary, a considerable complexity gap has remained, between NL hardness and NP membership, and that is what we close.

We believe this is noteworthy at least for the following reasons:

- To make progress on the challenge of the complexity of the general problem, it is natural to fix some parameters, especially the dimension. Bypassing the border between dimensions 2 and 3, which is where there is a jump beyond semi-linearity, seems to be very difficult with current techniques [cf. 1]. For dimension 1, the complexities were determined as NP-complete in the binary case [4] and NL-complete in the unary case [9].
- The unary encoding is used frequently enough, e.g. the classical modeling of concurrent systems by VASS [3] produces integers that are proportional to how many processes may interact in a single transition. Also, VASS given in unary can be translated without blow-up to *unary VASS* whose actions contain only -1, 0 and 1, and Lipton's lower bound holds already for such VASS.
- Our main result, that reachability for 2-VASS in unary is in NL, implies the PSPACE membership of the succinct variant. Moreover, and maybe most interestingly, we obtain the NL membership by proving that 2-VASS have reachability witnesses of *pseudo-polynomial* length, i.e. polynomial in the number of states and the maximum absolute value of any action, source or target integer. To our knowledge, this is the first time that the complexity of an interesting restriction of the reachability problem has broken 'the size of the reachability set barrier'. Namely, it is well-known that general VASS may have reachability sets which are finite but Ackermannianly large [2], and although some researchers conjecture that the reachability problem is primitive recursive or even of much smaller complexity, the Ackermann barrier remains. When the dimension is 2, it is not difficult to construct examples with exponentially large reachability sets (by employing weak doubling a number of times proportional to the number of states—this uses integers only up to absolute value 2), but we prove that polynomial reachability witnesses always exist.
- The technique we have developed seems novel, is surprisingly involved, and can be seen as a kind of extension of the classical 1-dimensional hill cutting [cf. e.g. 9] to dimension 2.

After a couple of preparatory sections, we present the main proof in Section 4, split into several stages. There, using the flattenings obtained by Blondin et al. [1], we are able to concentrate on obtaining short reachability witnesses for 2-VASS that are *LPSs*, i.e. without nested cycles. We then establish consequences for arbitrary 2-VASS in Section 5.

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2. On Our Marks

Here we recall, fix or introduce the basic notions, notations and problems we require.

Sets of Numbers. To restrict a set of numbers, we may write a condition in subscript, e.g. $\mathbb{N}_{\geq b}$ denotes the set of all non-negative integers that are at least b.

Lengths, Sizes and Norms. We denote the length or size by single bars, e.g. the length of a word w is written $|w|$.

To denote the infinity norm, we employ double bars. Thus, for a vector $\mathbf{v}$, $\|\mathbf{v}\|$ equals the maximum absolute value of any entry $\mathbf{v}_i$. Also, for a finite set $\mathbf{A}$ of vectors, $\|\mathbf{A}\|$ is the maximum of the infinity norms of its elements.

Rational Cones. We consider the cone spanned by a subset $\mathbf{C}$ of a d-dimensional rational space $\mathbb{Q}^d$ to be the closure of $\mathbf{C}$ under addition and under multiplication by positive rationals.

Note that the cone of $\mathbf{C}$ contains the zero vector only if it contains a line or one of the vectors in $\mathbf{C}$ is zero.

Paths and Admissibility. For a finite set $\mathbf{A} \subseteq \mathbb{Z}^d$, we have that vectors $\mathbf{a} \in \mathbf{A}$, finite words $\pi \in \mathbf{A}^*$ and languages $L \subseteq \mathbf{A}^*$ induce the following reachability relations on the d-dimensional non-negative integer space $\mathbb{N}^d$:

- $\mathbf{b} \xrightarrow{a} \mathbf{b}'$ iff $\mathbf{b} + \mathbf{a} = \mathbf{b}'$,
- $\xrightarrow{\pi} \stackrel{\text{def}}{=} \xrightarrow{\pi(1)}; \cdots ; \xrightarrow{\pi(|\pi|)}$, and
- $\xrightarrow{L} \stackrel{\text{def}}{=} \bigcup_{\pi \in L} \xrightarrow{\pi}$.

We often refer to a word $\pi \in \mathbf{A}^*$ as a *path*, and call the sum $\Sigma\pi \stackrel{\text{def}}{=} \pi(1) + \cdots + \pi(|\pi|)$ the *effect* of π. From a *source* $\mathbf{s} \in \mathbb{Z}^d$, the points *visited* by π are $\mathbf{s} + \pi(1) + \cdots + \pi(i)$ for all $i \in \{0, \ldots, |\pi|\}$, the last one being the *target* point. We say that π is *admissible* from $\mathbf{s}$ iff $\mathbf{s} \xrightarrow{\pi} \mathbf{t}$ for some $\mathbf{t}$, i.e., iff all the points visited are in $\mathbb{N}^d$, and also call π a path *from* $\mathbf{s}$ *to* $\mathbf{t}$ in this case.

Vector Addition Systems and Linear Path Schemes. We consider a d-dimensional *vector addition system with states* (d-VASS) to be a language over a finite alphabet $\mathbf{A} \subseteq \mathbb{Z}^d$ given by a non-deterministic finite automaton V.

A *linear path scheme* (LPS) is a special case when the language is given by a regular expression of the form

$$\Lambda = \alpha_0 \beta_1^* \alpha_1 \beta_2^* \cdots \beta_K^* \alpha_K$$

where all α_i and β_i are words in $\mathbf{A}^*$. We call $\beta_1, \ldots, \beta_K$ the *cycles* of Λ. Its *length* is $|\Lambda| \stackrel{\text{def}}{=} |\alpha_0 \beta_1 \alpha_1 \beta_2 \cdots \beta_k \alpha_k|$, and its *norm* $\|\Lambda\|$ is the maximum norm of any vector (i.e. letter) occuring in Λ.

Restricting further, we call Λ *simple* (an *S*LPS) when all α_i and β_i are of length 1, i.e., single vectors from $\mathbf{A}$.

Paths of Linear Path Schemes. We regard a path of an LPS as above to be given by a sequence of exponents, i.e. $n_1, \ldots, n_K$ where each n_i specifies how many times the cycle β_i is repeated in the path.

Note that several sequences of exponents may give the same word over $\mathbf{A}$. However, this non-uniqueness of representations will not cause difficulties.

Reachability Problems. These are the membership problems of the reachability relations that are induced by the VASS and LPS:

> Given a d-VASS V (resp., LPS Λ) and vectors $\mathbf{s}, \mathbf{t} \in \mathbb{N}^d$, decide whether $\mathbf{s} \xrightarrow{V} \mathbf{t}$ (resp., $\mathbf{s} \xrightarrow{\Lambda} \mathbf{t}$).

There are two variants of the problems: *unary* and *binary*, depending on how the integers in V (resp., Λ), $\mathbf{s}$ and $\mathbf{t}$ are encoded.

3. Get Set

We have six lemmas here that are useful in the sequel. The first four are essentially simple consequences in the plane of Cramer's Rule and Farkas-Minkowski-Weyl's Theorem.

From Cramer's Rule, we get that for cones that contain the zero vector, the latter is expressible using at most three vectors from the spanning set, moreover with small positive coefficients:

Lemma 1. *If the cone of $\mathbf{C} \subseteq^{\mathit{fin}} \mathbb{Z}^2$ contains $\mathbf{0}$, then $\mathbf{0}$ is a nonempty linear combination of at most three vectors from $\mathbf{C}$ and with coefficients in $\{1, \ldots, 2\|\mathbf{C}\|^2\}$.*

Furthermore, if $\mathbf{0}$ cannot be expressed like this with fewer than three vectors, the cone of $\mathbf{C}$ is equal to $\mathbb{Q}^2$.

Proof. If $\mathbf{C}$ contains $\mathbf{0}$, the statement is trivial. If $\mathbf{C}$ contains a vector $\mathbf{a}$ with a negative coordinate $\mathbf{a}_i$ as well as a vector $\mathbf{b} = -\lambda\mathbf{a}$ for some positive rational λ, then $\mathbf{0}$ can be expressed as $\mathbf{b}_i\mathbf{a} - \mathbf{a}_i\mathbf{b}$ and we are done. So now assume that $\mathbf{C}$ does not contain vectors $\mathbf{a}$ and $\mathbf{b}$ like this.

Consider a minimal subset $\mathbf{C}' \subseteq \mathbf{C}$ such that $\mathbf{0}$ can be expressed as a linear combination $\lambda_1\mathbf{a}^{(1)} + \cdots + \lambda_{|\mathbf{C}'|}\mathbf{a}^{(|\mathbf{C}'|)}$ with positive rational coefficients λ_i of vectors $\mathbf{a}^{(i)} \in \mathbf{C}'$. Assume for contradiction that $|\mathbf{C}'| > 3$. Then, there must be a closed half-plane containing at least 3 vectors, say w.l.o.g. $\mathbf{a}^{(1)}$, $\mathbf{a}^{(2)}$, and $\mathbf{a}^{(3)}$, from $\mathbf{C}'$. One of these three vectors can be expressed as a non-negative linear combination of the other two. Without loss of generality assume $\mathbf{a}^{(1)} = c_1\mathbf{a}^{(2)} + c_2\mathbf{a}^{(3)}$ with $c_1, c_2 \geq 0$. But then we can write

$$\mathbf{0} = \lambda_1(c_1\mathbf{a}^{(2)} + c_2\mathbf{a}^{(3)}) + \lambda_2\mathbf{a}^{(2)} + \lambda_3\mathbf{a}^{(3)} + \cdots + \lambda_{|\mathbf{C}'|}\mathbf{a}^{(|\mathbf{C}'|)}$$

and express $\mathbf{0}$ as a linear combination with positive coefficients of only $|\mathbf{C}'| - 1$ vectors contradicting the minimality of $\mathbf{C}'$.

Therefore we can choose three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{C}$ such that there are strictly positive x_1, x_2, x_3 and $x_1\mathbf{a} + x_2\mathbf{b} + x_3\mathbf{c} = \mathbf{0}$.

The equation has infinitely many solution since we can scale the coefficients. However, if we set x_3 to be, say, $|\mathbf{b}_1\mathbf{a}_2 - \mathbf{a}_1\mathbf{b}_2|$ the solution becomes unique (since $\mathbf{a}$ and $\mathbf{b}$ are linearly independent) and it can be easily checked that the solution obtained by Cramer's rule is $x_1 = |\mathbf{c}_1\mathbf{b}_2 - \mathbf{b}_1\mathbf{c}_2|$ and $x_2 = |\mathbf{a}_1\mathbf{c}_2 - \mathbf{c}_1\mathbf{a}_2|$.

For the second statement of the lemma observe that we can express $-\mathbf{a}$ and $-\mathbf{b}$ as linear combinations of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ with positive rationals. For example, $-\mathbf{a} = (x_2\mathbf{b} + x_3\mathbf{c})/x_1$. Since $\mathbf{a}$ and $\mathbf{b}$ are linearly independent, any vector in $\mathbb{Q}^2$ can be expressed as a linear combination of $\mathbf{a}$ and $\mathbf{b}$ using rational coefficients. Combined with the fact that we can express $-\mathbf{a}$ and $-\mathbf{b}$ the claim follows. ☐

The next two lemmas apply to the other case, i.e. when the cone does not contain the zero vector: firstly, such cones are determined by pairs of outermost vectors in their spanning sets; and secondly, they are contained in open halfplanes determined by small vectors.

Let us write $\mathbf{v}_{\circlearrowright} \stackrel{\text{def}}{=} \langle \mathbf{v}_2, -\mathbf{v}_1 \rangle$ and $\mathbf{v}_{\circlearrowleft} \stackrel{\text{def}}{=} \langle -\mathbf{v}_2, \mathbf{v}_1 \rangle$ for the vector $\mathbf{v} \in \mathbb{Z}^2$ rotated 90° clockwise and anticlockwise, respectively.

Lemma 2. *If the cone of $\emptyset \neq \mathbf{C} \subseteq \mathbb{Z}^2$ does not contain $\mathbf{0}$, then there are two vectors $\mathbf{a}, \mathbf{b} \in \mathbf{C}$ such that $\{\mathbf{a}, \mathbf{b}\}$ spans the same cone as $\mathbf{C}$, and for all $\mathbf{c}$ in the cone of $\mathbf{C}$, $\mathbf{a}_{\circlearrowleft} \cdot \mathbf{c} \geq 0$ and $\mathbf{b}_{\circlearrowright} \cdot \mathbf{c} \geq 0$.*

Proof. Consider a subset $\mathbf{C}' \subseteq \mathbf{C}$ of minimum size that spans the same cone as $\mathbf{C}$. Assume for contradiction that $|\mathbf{C}'| > 2$. Then the set contains three vectors $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ and because these vectors must be linearly dependent we have $\mathbf{z} = \lambda_1\mathbf{x} + \lambda_2\mathbf{y}$ for some rationals λ_1 and λ_2. We can assume, without loss of generality, that λ_1 and λ_2 do not have different signs (otherwise we can appropriately rename $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$). If λ_1 and λ_2 are non-negative, $\mathbf{C}' \setminus \{\mathbf{z}\}$ still spans the same cone as $\mathbf{C}'$ since in any positive combination, $\mathbf{z}$ can be replaced by $\lambda_1\mathbf{x}+\lambda_2\mathbf{y}$. If however, λ_1 and λ_2 are non-positive, the cone spanned by $\mathbf{C}'$ contains $\mathbf{0}$ since $\mathbf{0}=\mathbf{z}-\lambda_1\mathbf{x}-\lambda_2\mathbf{y}$. In both cases we get a contradiction to our assumptions.

So there must indeed be two vectors $\mathbf{a},\mathbf{b}\in\mathbf{C}$, not necessarily different, that span the same cone as $\mathbf{C}$. Observe that $\mathbf{b}\cdot\mathbf{a}_{\circlearrowleft}<0 \iff \mathbf{b}\cdot\mathbf{a}_{\circlearrowright}>0$ because $\mathbf{a}_{\circlearrowright}=-\mathbf{a}_{\circlearrowleft}$. Further observe that $\mathbf{b}\cdot\mathbf{a}_{\circlearrowright}=\mathbf{b}_{\circlearrowleft}\cdot\mathbf{a}$. Therefore, either $\mathbf{b}\cdot\mathbf{a}_{\circlearrowleft}\geq 0$ or $\mathbf{b}\cdot\mathbf{a}_{\circlearrowright}=\mathbf{b}_{\circlearrowleft}\cdot\mathbf{a}\geq 0$ holds. We assume w.l.o.g. that $\mathbf{b}\cdot\mathbf{a}_{\circlearrowleft}\geq 0$, since otherwise we can swap the names of $\mathbf{a}$ and $\mathbf{b}$.

Pick any $\mathbf{c}\in\mathbf{C}$. Since the cone of $\{\mathbf{a},\mathbf{b}\}$ contains $\mathbf{c}$, there exist $x,y\geq 0$ such that $\mathbf{c}=x\mathbf{a}+y\mathbf{b}$ and therefore

$$\mathbf{c}\cdot\mathbf{a}_{\circlearrowleft}=(x\mathbf{a}+y\mathbf{b})\cdot\mathbf{a}_{\circlearrowleft}=x\mathbf{a}\cdot\mathbf{a}_{\circlearrowleft}+y\mathbf{b}\cdot\mathbf{a}_{\circlearrowleft}\geq 0,$$

because $\mathbf{a}x\cdot\mathbf{a}_{\circlearrowleft}=0$. Analogously, using $y\mathbf{b}\cdot\mathbf{b}_{\circlearrowright}=0$, we get

$$\mathbf{c}\cdot\mathbf{b}_{\circlearrowright}=(x\mathbf{a}+y\mathbf{b})\cdot\mathbf{b}_{\circlearrowright}=x\mathbf{a}\cdot\mathbf{b}_{\circlearrowright}+y\mathbf{b}\cdot\mathbf{b}_{\circlearrowright}\geq 0. \qquad \square$$

Lemma 3. *If the cone of $\emptyset\neq\mathbf{C}\subseteq^{fin}\mathbb{Z}^2$ does not contain $\mathbf{0}$, then there exists a vector $\mathbf{p}\in\mathbb{Z}^2$ such that $\|\mathbf{p}\|\leq 2\|\mathbf{C}\|$ and $\mathbf{p}\cdot\mathbf{c}>0$ for all $\mathbf{c}\in\mathbf{C}$.*

Proof. According to Lemma 2 we have vectors $\mathbf{a},\mathbf{b}\in\mathbf{C}$ such that $\{\mathbf{a},\mathbf{b}\}$ spans the same cone as $\mathbf{C}$, and for all $\mathbf{c}\in\mathbf{C}$, $\mathbf{a}_{\circlearrowleft}\cdot\mathbf{c}\geq 0$ and $\mathbf{b}_{\circlearrowright}\cdot\mathbf{c}\geq 0$.

If $\mathbf{a}$ alone already spans the same cone as $\mathbf{C}$, we can choose $\mathbf{p}=\mathbf{a}$ and are done. Otherwise, $\mathbf{a}$ and $\mathbf{b}$ are linearly independent and we choose $\mathbf{p}=\mathbf{a}_{\circlearrowleft}+\mathbf{b}_{\circlearrowright}$. Clearly $\|\mathbf{p}\|\leq 2\|\mathbf{C}\|$. For any $\mathbf{c}\in\mathbf{C}$, $\mathbf{p}\cdot\mathbf{c}=(\mathbf{a}_{\circlearrowleft}+\mathbf{b}_{\circlearrowright})\cdot\mathbf{c}$. Since $\mathbf{a}$ and $\mathbf{b}$ are linearly independent, $\mathbf{a}_{\circlearrowleft}\cdot\mathbf{c}\neq 0$ or $\mathbf{b}_{\circlearrowright}\cdot\mathbf{c}\neq 0$ and therefore $(\mathbf{a}_{\circlearrowleft}+\mathbf{b}_{\circlearrowright})\cdot\mathbf{c}>0$. $\square$

Our last lemma dealing with cones gives some additional properties for the structure of the cones when it is known that the cone does not contain some vector. For simplicity, and because it is all we will need later, we focus on the case that $\langle 0,1\rangle$ is not contained in the cone.

Lemma 4. *Let $\emptyset\neq\mathbf{C}\subseteq^{fin}\mathbb{Z}^2$ be a set not containing $\mathbf{0}$. If the cone of $\mathbf{C}$ does not contain $\langle 0,1\rangle$, then there is a vector $\mathbf{p}\in\mathbb{Z}^2$ such that*

- $\|\mathbf{p}\|\leq\|\mathbf{C}\|$,
- $\mathbf{p}\cdot\langle 0,1\rangle<0$,
- $\mathbf{p}\cdot\mathbf{c}\geq 0$ *for all* $\mathbf{c}\in\mathbf{C}$, *and*
- *if* $\mathbf{p}_1<0$, *then* $\mathbf{p}_{\circlearrowright}\in\mathbf{C}$.

Proof. We distinguish two basic cases based on whether the cone of $\mathbf{C}$ contains $\mathbf{0}$ or not. First suppose the cone of $\mathbf{C}$ does not contain $\mathbf{0}$. Then, by Lemma 2, there are vectors $\mathbf{a},\mathbf{b}\in\mathbf{C}$ such that $\{\mathbf{a},\mathbf{b}\}$ spans the same cone as $\mathbf{C}$, and for all $\mathbf{c}\in\mathbf{C}$, $\mathbf{a}_{\circlearrowleft}\cdot\mathbf{c}\geq 0$ and $\mathbf{b}_{\circlearrowright}\cdot\mathbf{c}\geq 0$. Note that, in particular, we can plug in $\mathbf{b}$ for $\mathbf{c}$ and then must have $\mathbf{a}_{\circlearrowleft}\cdot\mathbf{b}\geq 0$ which implies $\mathbf{a}_1\mathbf{b}_2-\mathbf{b}_1\mathbf{a}_2\geq 0$.

There are three candidates for the choice of $\mathbf{p}$: $\mathbf{a}_{\circlearrowleft}$, $\langle 0,-1\rangle$, and $\mathbf{b}_{\circlearrowright}$. Suppose $\mathbf{p}=\mathbf{a}_{\circlearrowleft}$ does not satisfy all conditions of the lemma. Then we must have $\mathbf{a}_{\circlearrowleft}\cdot\langle 0,1\rangle\geq 0$ and therefore $\mathbf{a}_1\geq 0$. Assume further that $\langle 0,-1\rangle$ also does not satisfy all conditions of the lemma. Then there must be a vector $\mathbf{c}'\in\mathbf{C}$ with $\mathbf{c}'_2>0$.

We now show that if neither $\mathbf{a}_{\circlearrowleft}$ nor $\langle 0,-1\rangle$ can be used for $\mathbf{p}$, $\mathbf{b}_{\circlearrowright}$ can. Assume for contradiction that $\mathbf{b}_{\circlearrowright}\cdot\langle 0,1\rangle\geq 0$. But then $\mathbf{b}_1\leq 0$ and we can express $\langle 0,\mathbf{a}_1\mathbf{b}_2-\mathbf{b}_1\mathbf{a}_2\rangle=\mathbf{a}_1\mathbf{b}-\mathbf{b}_1\mathbf{a}$ as a positive combination of $\mathbf{a}$ and $\mathbf{b}$. Since the cone of $\mathbf{C}$ does not contain $\mathbf{0}$ or $\langle 0,1\rangle$ we must have $\mathbf{a}_1\mathbf{b}_2-\mathbf{b}_1\mathbf{a}_2<0$ which, as we argued above, cannot be the case. Here we used the assumption that we do not have $\mathbf{a}_1=\mathbf{b}_1=0$. If that were the case, either $\langle 0,1\rangle$ would be in the cone (if $\mathbf{a}_2>0$ or $\mathbf{b}_2>0$) or $\mathbf{c}'$ could not be in the cone spanned by $\mathbf{a}$ and $\mathbf{b}$, which is a contradiction.

We conclude that $\mathbf{b}_{\circlearrowright}\cdot\langle 0,1\rangle<0$ and thus $\mathbf{b}_1>0$. To finish the proof we have to argue that $\mathbf{p}=\mathbf{b}_{\circlearrowright}$ is a valid choice and we do this by showing that $\mathbf{p_1}=\mathbf{b}_2\geq 0$. Assume for contradiction that $\mathbf{b}_2<0$. Since $\mathbf{a}_1\geq 0$ and $\mathbf{b}_1\mathbf{a}_2\leq\mathbf{a}_1\mathbf{b}_2\leq 0$, we can conclude that $\mathbf{a}_2\leq 0$. However, if $\mathbf{b}_2<0$ and $\mathbf{a}_2\leq 0$, $\mathbf{c}'$ cannot be in the cone spanned by $\mathbf{a}$ and $\mathbf{b}$, which is a contradiction.

We now move to the second case in which we assume that the cone of $\mathbf{C}$ does contain $\mathbf{0}$. Then there are two vectors $\mathbf{a},\mathbf{b}\in\mathbf{C}$ such that $\mathbf{a}+\lambda\mathbf{b}=\mathbf{0}$ for some positive rational λ. This is because if $\mathbf{0}$ could only be expressed with three or more vectors, according to Lemma 1, $\langle 0,1\rangle$ would also be in the cone of $\mathbf{C}$.

Clearly, either $\mathbf{a}_1\leq 0$ or $\mathbf{b}_1\leq 0$. Without loss of generality let $\mathbf{a}_1\leq 0$. We choose $\mathbf{p}=\mathbf{a}_{\circlearrowleft}$.

The only condition of the lemma not trivially met is that $\mathbf{p}\cdot\mathbf{c}\geq 0$ for all $\mathbf{c}\in\mathbf{C}$. Assume that there is a $\mathbf{c}\in\mathbf{C}$ such that $\mathbf{a}_{\circlearrowleft}\cdot\mathbf{c}<0$. Then $\mathbf{c}_1\mathbf{a}_2-\mathbf{a}_1\mathbf{c}_2>0$. If $\mathbf{c}_1\geq 0$, $\langle 0,1\rangle$ would be in the cone of $\mathbf{C}$ since it can be expressed as $\mathbf{c}_1\cdot\mathbf{a}-\mathbf{a}_1\cdot\mathbf{c}/(\mathbf{c}_1\mathbf{a}_2-\mathbf{a}_1\mathbf{c}_2)$. Otherwise $\langle 0,1\rangle$ would also be in the cone of $\mathbf{C}$ since it can be expressed as $\mathbf{b}_1\cdot\mathbf{c}-\mathbf{c}_1\cdot\mathbf{b}/(\mathbf{b}_1\mathbf{c}_2-\mathbf{c}_1\mathbf{b}_2)$. Either way, we have a contradiction. $\square$

Moving from rational cones to paths of SLPSs, our remaining two lemmas pin down some relatively basic properties of SLPS paths in which some cycles are repeated 'many' times: firstly, if all those cycles are contained in a halfplane, then the effect of the path must point roughly in the same direction (we have a strict and a non-strict version here); secondly, if the path when started at a point remains sufficiently far from both axes (i.e. respects a sufficiently wide margin), then it can be shortened admissibly by a range of multiples of any small vector that is in the cone spanned by the 'often' repeated cycles.

For a path π of a 2-SLPS $\Lambda=\alpha_0\beta_1^*\alpha_1\beta_2^*\ldots\beta_K^*\alpha_K$ and a bound $B\in\mathbb{N}$, let

$$\mathrm{Cycles}_{\geq B}(\Lambda,\pi)\subseteq\mathbb{Z}^2$$

be the set of all cycles of Λ that are repeated in π at least B times.

Lemma 5. *Suppose π is a path of a 2-SLPS Λ with K cycles, $B\in\mathbb{N}$ and $\mathbf{p}\in\mathbb{Z}^2$.*

(i) *If $\mathbf{p}\cdot\mathbf{a}>0$ for all $\mathbf{a}\in\mathrm{Cycles}_{\geq B}(\Lambda,\pi)$, then*

$$\mathbf{p}\cdot\Sigma\pi\ \geq\ |\pi|-(KB+1)(2\|\Lambda\|\,\|\mathbf{p}\|+1).$$

(ii) *If $\mathbf{p}\cdot\mathbf{a}\geq 0$ for all $\mathbf{a}\in\mathrm{Cycles}_{\geq B}(\Lambda,\pi)$, then*

$$\mathbf{p}\cdot\Sigma\pi\ \geq\ -(KB+1)(2\|\Lambda\|\,\|\mathbf{p}\|).$$

Proof. The effect of π can be decomposed as $\Sigma\pi=\mathbf{v}+\mathbf{b}$, where $\mathbf{v}$ is the combined effect of those cycles occurring at least B times and $\mathbf{b}$ is the rest. Hence $\mathbf{v}$ is a linear combination $\mathbf{v}=\sum_{i=1}^{\ell}\mathbf{a}^{(i)}$, where $\mathbf{a}^{(i)}\in\mathrm{Cycles}_{\geq B}(\Lambda,\pi)$ and $\mathbf{b}$ is the effect of a path of length $|\pi|-\ell\leq KB+1$. We can therefore estimate

$$\mathbf{p}\cdot\mathbf{b}\geq -2(KB+1)\|\Lambda\|\|\mathbf{p}\|. \tag{1}$$

If $\mathbf{p}\cdot\mathbf{a}>0$ for all $\mathbf{a}\in\mathrm{Cycles}_{\geq B}(\Lambda,\pi)$ then

$$\mathbf{p}\cdot\mathbf{v}=\sum_{i=1}^{\ell}\mathbf{p}\cdot\mathbf{a}^{(i)}\ \geq\ \ell\ \geq\ |\pi|-(KB+1). \tag{2}$$

The first claim therefore follows by Equations 1 and 2 and by the fact that $\mathbf{p}\cdot\Sigma\pi=\mathbf{p}\cdot\mathbf{v}+\mathbf{p}\cdot\mathbf{b}$.

For the second claim, just observe that if $\mathbf{p}\cdot\mathbf{a}\geq 0$ for all $\mathbf{a}\in\mathrm{Cycles}_{\geq B}(\Lambda,\pi)$, then $\mathbf{p}\cdot\mathbf{v}=\sum_{i=1}^{l}\mathbf{p}\cdot\mathbf{a}^{(i)}\geq 0$. $\square$

Let us call a path π' a *shortening of path π by vector* $\mathbf{e}$ when π' is a proper subword (not necessarily contiguous) of π and $\Sigma\pi'=\Sigma\pi-\mathbf{e}$.

Lemma 6. *Suppose a path π of a 2-SLPS Λ, $N\in\mathbb{N}$, $\mathbf{c}\in\mathbb{Z}^2$ and $\mathbf{s}\in\mathbb{N}^2$ satisfy:*

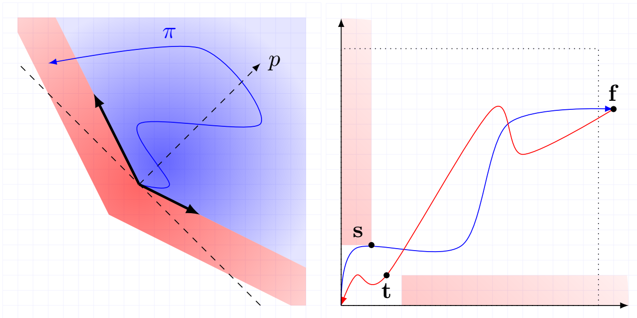

Figure 1. Lemma 5 (left): The path π must remain in the red/blue area. In case (ii) the red belt is parallel to the dashed line, i.e. orthogonal to $\mathbf{p}$. Theorem 11 (right): The path from $\mathbf{s}$ to $\mathbf{t}$ via a sufficiently large point $\mathbf{f}$ can be shortened.

- $\|\Lambda\| > 0$ *and* $\|\mathbf{c}\| \leq \|\Lambda\|$,
- *the cone of* $\mathrm{Cycles}_{\geq 2\|\Lambda\|^2 N}(\Lambda, \pi)$ *contains* $\mathbf{c}$*, and*
- *all points visited by* π *from* $\mathbf{s}$ *are in* $(\mathbb{N}_{\geq 6\|\Lambda\|^3 N})^2$.

There exists $\gamma \in \{1, \ldots, 2\|\Lambda\|^2\}$ *such that, for all* $n \in \{1, \ldots, N\}$, π *has a shortening by* $n\gamma\mathbf{c}$ *which is admissible from* $\mathbf{s}$.

Proof. Let $\mathbf{C} \stackrel{\text{def}}{=} \mathrm{Cycles}_{\geq 2\|\Lambda\|^2 N}(\Lambda, \pi)$. We claim that

$$\gamma\mathbf{c} = \lambda_1 \mathbf{a}^{(1)} + \cdots + \lambda_j \mathbf{a}^{(j)}$$

for some $j \in \{1, 2, 3\}$, $\mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(j)} \in \mathbf{C}$ and $\gamma, \lambda_1, \ldots, \lambda_j \in \{1, \ldots, 2\|\Lambda\|^2\}$. If $\mathbf{c} = \mathbf{0}$, this directly follows from Lemma 1. Otherwise, reasoning as in the proof of Lemma 2, there must be two vectors $\mathbf{a}^{(1)}, \mathbf{a}^{(2)} \in \mathbf{C}$ such that the cone spanned by $\{\mathbf{a}^{(1)}, \mathbf{a}^{(2)}\}$ contains $\mathbf{c}$ but not $\mathbf{0}$. Then the claim follows by Lemma 1 applied to the set $\{-\mathbf{c}, \mathbf{a}^{(1)}, \mathbf{a}^{(2)}\}$.

Now, we can subtract $n\gamma\mathbf{c}$ from the effect of π by deleting $n\lambda_i$ occurences of the cycle $\mathbf{a}^{(i)}$ for all $i \in \{1, \ldots, j\}$. Any such shortening π' is admissible from $\mathbf{s}$ because, for any point visited by π, the differences between its coordinates and the coordinates of the corresponding point visited by π' are at most $6\|\Lambda\|^3 n$. □

4. Go!

Here is the bulk of our work.

We present a sequence of theorems that culminates in Theorem 12, which establishes that if a reachability witness of a 2-dimensional simple linear path scheme cannot be shortened, then it cannot visit points whose norm exceeds a certain polynomial bound (in the length and the norm of the SLPS).

A key step towards the last theorem is Theorem 11, where lemmas from the previous section are employed to conclude that it suffices to prove that shortest reachability witnesses cannot visit points that are 'near' one of the axes but further from the other axis than a certain polynomial bound (smaller than the one in Theorem 12, see the red margins in Figure 1 on the right).

The remainder of our reasoning here is therefore concerned with showing that shortest reachability witnesses cannot contain points that are, without loss of generality, within a y-axis margin but too far from the x-axis (more than a polynomial bound). We accomplish this by proving that, if such a scenario occurs, then we can focus on a point $\mathbf{t}$ that is within the y-axis margin and maximally far from the x-axis, and find an admissible shortening of the reachability witness whose effect on $\mathbf{t}$ is to decrease its y-coordinate by a 'small' amount.

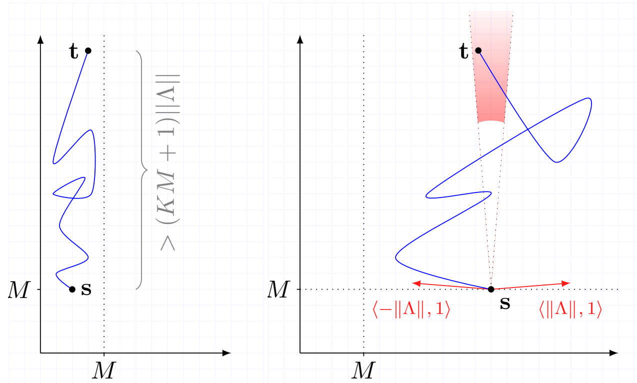

Figure 2. Illustrations of Theorem 7 (left) and Theorem 8 (on the right, with $M \stackrel{\text{def}}{=} 6\|\Lambda\|^3 N$).

Theorems 7–10 provide increasingly powerful tools for identifying admissible shortenings of paths that in some way climb the y-axis. In the proof of Theorem 12, such shortenings are applied to appropriate segments and reversals of segments of reachability witnesses. Thus their effects have to be matched (recall 1-dimensional hill cutting [cf. e.g. 9]), which explains the ranges of possible shortenings in Theorems 7–10.

We begin with handling the case in which a path goes up by a large amount but only visits points which are close to the y-axis and not close to the x-axis, cf. Figure 2 on the left.

Recall that, for planar vectors $\mathbf{v}$, we denote their horizontal and vertical components by $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively.

Theorem 7. *Suppose a 2-SLPS* Λ *with at most* K *cycles has a path* π *from point* $\mathbf{s}$ *to point* $\mathbf{t}$ *such that for some* $M \in \mathbb{N}$

- *all points visited by* π *from* $\mathbf{s}$ *are in* $\mathbb{N}_{<M} \times \mathbb{N}_{\geq M}$ *and*
- $(\mathbf{t} - \mathbf{s})_2 > (KM + 1)||\Lambda||$.

There is $\gamma \in \{1, \ldots, ||\Lambda||\}$ *such that, for all* $n \in \{1, \ldots, \lfloor M/\gamma \rfloor\}$, π *has a shortening by* $\langle 0, n\gamma\rangle$ *which is admissible from* $\mathbf{s}$.

Proof. There is a cycle $\mathbf{c}$ in π that is repeated at least M times. Otherwise, for the effect of π, $||\mathbf{t} - \mathbf{s}|| \leq ((K+1) + K \cdot (M-1)) \cdot ||\Lambda|| = (KM+1)||\Lambda||$, which contradicts the second assumption of the theorem. Let $\mathbf{u}$ and $\mathbf{v}$ be the points visited right before the first, and right after the last repetitions of the cycle $\mathbf{c}$, respectively. The first coordinate of $\mathbf{c}$ is 0 since otherwise $|(\mathbf{u}-\mathbf{v})_1| \geq M$, which contradicts the first assumption of the theorem. Therefore $\mathbf{c} = \langle 0, \gamma\rangle$ for some $\gamma \in \{1, \ldots, ||\Lambda||\}$ and thus, π has a shortening by $\langle 0, n\gamma\rangle$ for all $n \in \{1, \ldots, \lfloor M/\gamma \rfloor\}$. This shortening is admissible since it does not affect the first coordinate of any point visited, only decreases the second coordinates by at most $\lfloor M/\gamma \rfloor \cdot \gamma \leq M$, and all visited points have a second coordinate value of at least M prior to the shortening. □

The following theorem deals with a case in which all points visited on a path are far from both axes but where the total effect of the path is much bigger in the second coordinate than the first, cf. Figure 2 on the right, where $M = 6||\Lambda||^3 N$.

Theorem 8. *Suppose a 2-SLPS* Λ *with at most* K *cycles has a path* π *from point* $\mathbf{s}$ *to point* $\mathbf{t}$ *such that for some* $N \in \mathbb{N}$

- *all points visited by* π *from* $\mathbf{s}$ *are in* $(\mathbb{N}_{\geq 6||\Lambda||^3 N})^2$ *and*
- *for all* $\lambda \in [-||\Lambda||, ||\Lambda||]$, $\langle \lambda, 1\rangle \cdot (\mathbf{t} - \mathbf{s}) > (4KN + 2)||\Lambda||^4$.

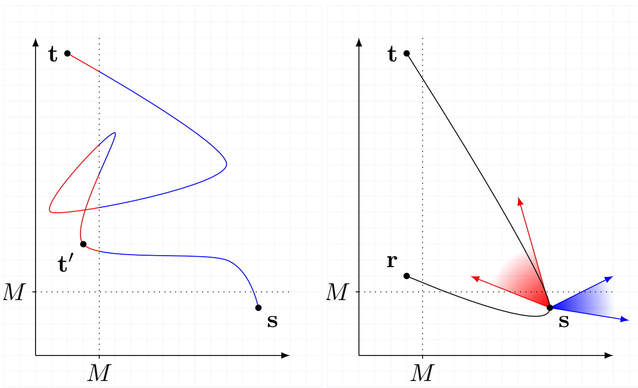

Figure 3. Theorem 9 (left): Either the cone of cycles in the segment from $\mathbf{t}'$ to $\mathbf{t}$ contains $\langle 0,1\rangle$ or that from $\mathbf{s}$ to $\mathbf{t}'$ contains some short vector in the top-left quadrant. Theorem 10 (right): The cone of cycles from $\mathbf{r}$ to $\mathbf{s}$ (blue), combined with the cone of cycles from $\mathbf{s}$ to $\mathbf{t}$ (red), contains $\langle 0,1\rangle$.

$\langle 0,1\rangle$ is in the cone of $\mathrm{Cycles}_{\geq 2||\Lambda||^2 N}(\Lambda,\pi)$ *and there exists $\gamma \in \{1,\ldots,2||\Lambda||^2\}$ such that, for all $n \in \{1,\ldots,N\}$, π has a shortening by $\langle 0,n\gamma\rangle$ which is admissible from* $\mathbf{s}$.

Proof. Note that $(\mathbf{t}-\mathbf{s})_2 > 0$, since otherwise either $\langle 1,1\rangle\cdot(\mathbf{t}-\mathbf{s})$ or $\langle -1,1\rangle\cdot(\mathbf{t}-\mathbf{s})$ would be non-positive contradicting the assumption of the theorem.

Let $\mathbf{C} = \mathrm{Cycles}_{\geq 2||\Lambda||^2 N}(\Lambda,\pi)$. Assume for contradiction that $\langle 0,1\rangle$ is not in the cone of $\mathbf{C}\setminus\{\mathbf{0}\}$. Then, due to Lemma 4, there exists $\mathbf{p}\in\mathbb{Z}^2$ such that $||\mathbf{p}|| \leq ||\Lambda||$, $\mathbf{p}\cdot\langle 0,1\rangle < 0$, and $\mathbf{p}\cdot\mathbf{a} \geq 0$ for all $\mathbf{a}\in\mathbf{C}$. This implies $\mathbf{p}_2 < 0$ and therefore

$$-\mathbf{p}\cdot(\mathbf{t}-\mathbf{s}) \geq \langle -\mathbf{p}_1, 1\rangle\cdot(\mathbf{t}-\mathbf{s}) > (4KN+2)||\Lambda||^4.$$

But, by Lemma 5 (ii),

$$\begin{aligned}\mathbf{p}\cdot(\mathbf{t}-\mathbf{s}) &\geq -(K2\|\Lambda\|^2 N+1)(2\|\Lambda\|^2)\\ &\geq -(4KN+2)\|\Lambda\|^4.\end{aligned}$$

Therefore, $\langle 0,1\rangle$ must be in the cone of $\mathbf{C}$, and we conclude by Lemma 6. ☐

In the next theorem we combine the previous two results to handle the case when a path starts close to the x-axis, ends close to the y-axis but far away from the x-axis and does not come close the the x-axis anywhere in between, cf. Figure 3 on the left.

Theorem 9. *Suppose $N\in\mathbb{N}$, $M \geq 6\|\Lambda\|^3 N$, and a 2-SLPS Λ with $K>0$ cycles has a path π from point* $\mathbf{s}$ *to point* $\mathbf{t}$ *such that*

- $\mathbf{s}_1 \geq 0$, $\mathbf{s}_2 < M$,
- $\mathbf{t}_1 < M$, $\mathbf{t}_2 \geq 12(K+1)(M+1)||\Lambda||^4$ *and*
- *all points visited by π after* $\mathbf{s}$ *are in $\mathbb{N}\times\mathbb{N}_{\geq M}$.*

Let π' be the shortest nonempty prefix of π whose target point $\mathbf{t}'$ satisfies ${\mathbf{t}'}_1 < M$. Provided $|\pi'| \geq 2$, let $\pi^\dagger$ be π' without its first and last vectors, let $\Lambda^\dagger$ be an SLPS one of whose paths is $\pi^\dagger$ and whose length and norm are at most those of Λ, and let $\mathbf{C} = \mathrm{Cycles}_{\geq 2||\Lambda||^2 N}(\Lambda^\dagger,\pi^\dagger)$.

(i) *If either $(\mathbf{t}-\mathbf{t}')_2 > 6(K+1)(M+1)||\Lambda||^4$ or $\langle 0,1\rangle$ is in the cone of* $\mathbf{C}$, *then there exists $\gamma\in\{1,\ldots,2||\Lambda||^2\}$ such that, for all $n\in\{1,\ldots,N\}$, π has a shortening by $\langle 0,n\gamma\rangle$ which is admissible from* $\mathbf{s}$.

(ii) *Otherwise, there exists* $\mathbf{v}\in\mathbf{C}\cap(\mathbb{Z}_{<0}\times\mathbb{Z}_{>0})$ *such that*

$$\mathbf{v}_{\circlearrowleft}\cdot\langle\mathbf{s}_1,-\mathbf{t}_2\rangle < 7(K+2)(M+1)||\Lambda||^5.$$

Proof. If $(\mathbf{t}-\mathbf{t}')_2 > 6(K+1)(M+1)||\Lambda||^4$, let π'' be the rest of π after π', i.e., the segment of π that starts at $\mathbf{t}'$ and ends at $\mathbf{t}$. Then partition π'' into segments that visit only points in $\mathbb{N}_{<M}\times\mathbb{N}$ and segments for which all intermediate points are outside that set. Call these segments y-axis-close and y-axis-far, respectively. In the following we argue that either Theorem 7 applies to one of the former segments, or Theorem 8 applies to one of the latter segments.

Let ℓ be the total number of segments and, for $i\in[1,\ell-1]$, let $\mathbf{a}^{(i)}$ be the endpoint of the i-th segment and the start point of the $(i+1)$-th segment. Note that a path from an SLPS with at most K cycles will be split into at most $2(K+1)$ segments and therefore $\ell \leq 2(K+1)$. For convenience, define $\mathbf{a}^{(0)}$ to be $\mathbf{t}'$ and $\mathbf{a}^{(\ell)}$ to be $\mathbf{t}$.

Each segment corresponds to a SLPS that is a fragment of the original SLPS. Let the SLPS fragment of the i-th segment contain K_i cycles. Note that each of the cycles in the original SLPS can only be part of two different segments. Therefore, $\sum K_i \leq 2K$. Since

$$\begin{aligned}&\sum_{i=1}^{\ell}(\mathbf{a}^{(i)}-\mathbf{a}^{(i-1)})_2 = (\mathbf{t}-\mathbf{t}')_2\\ &> 6(K+1)(M+1)||\Lambda||^4\\ &> 2KM||\Lambda|| + 2(K+1)(M+1)||\Lambda|| + 4(K+1)||\Lambda||^4,\end{aligned}$$

there must be a segment i, going from $\mathbf{a}^{(i-1)}$ to $\mathbf{a}^{(i)}$, for which

$$(\mathbf{a}^{(i)}-\mathbf{a}^{(i-1)})_2 > (K_iM+M+1)||\Lambda|| + 2||\Lambda||^4.$$

If this segment i is y-axis-close, we observe that $(\mathbf{a}^{(i)}-\mathbf{a}^{(i-1)})_2 > (K_iM+1)||\Lambda||$ and therefore Theorem 7 applies to it.

If this segment i is y-axis-far then

$$\begin{aligned}(\mathbf{a}^{(i)}-\mathbf{a}^{(i-1)})_2 &> (K_i6N||\Lambda||^3+M+1)||\Lambda|| + 2||\Lambda||^4\\ &> (K_i4N+2)||\Lambda||^4 + 2||\Lambda|| + M||\Lambda||,\end{aligned}$$

since $M \geq 6||\Lambda||^3 N$.

Now consider the point ${\mathbf{a}^{(i-1)}}'$ visited right after $\mathbf{a}^{(i-1)}$ and the point ${\mathbf{a}^{(i)}}'$ visited right before $\mathbf{a}^{(i)}$ and consider the path between $\mathbf{a}^{(i-1)}$ and $\mathbf{a}^{(i)}$ without the first and last vector. Note that ${\mathbf{a}_1^{(i-1)}}', {\mathbf{a}_1^{(i)}}' \in [M, M+||\Lambda||)$ and hence $|({\mathbf{a}^{(i)}}' - {\mathbf{a}^{(i-1)}}')_1| \leq ||\Lambda|| < M$. Therefore, we have $\langle\lambda,1\rangle\cdot({\mathbf{a}^{(i)}}' - {\mathbf{a}^{(i-1)}}') > (K_i4N+2)||\Lambda||^4$ for all $\lambda\in[-||\Lambda||,||\Lambda||]$ and hence Theorem 8 applies to this subpath, going from ${\mathbf{a}^{(i-1)}}'$ to ${\mathbf{a}^{(i)}}'$.

Note that the section of π going from $\mathbf{s}$ to $\mathbf{a}^{(i-1)}$ (or ${\mathbf{a}^{(i-1)}}'$, respectively) is still admissible after the shortening carried out through Theorem 7 or Theorem 8. The shortened segment i is also admissible due to these theorems. The section of π that started at $\mathbf{a}^{(i)}$ prior to the shortening is also admissible since the first coordinate of the corresponding points is not changed and the second coordinate is decreased by at most $N2||\Lambda||^2 < M$. Moreover, the second coordinate of all the points prior to the shortening was at least M.

In the remainder of the proof, assume

$$(\mathbf{t}-\mathbf{t}')_2 \leq 6(K+1)(M+1)||\Lambda||^4$$

and consequently

$${\mathbf{t}'}_2 \geq 6(K+1)(M+1)||\Lambda||^4,$$

since $\mathbf{t}_2 \geq 12(K+1)(M+1)||\Lambda||^4$.

Then $|\pi'| \geq 2$, so $\pi^\dagger$, $\Lambda^\dagger$ and $\mathbf{C}$ are well defined. Let $\mathbf{s}^\dagger$ be the first point visited by π' after $\mathbf{s}$, and let $\mathbf{t}^\dagger$ be the target point of $\pi^\dagger$ from $\mathbf{s}^\dagger$. Observe, that ${\mathbf{s}^\dagger}_1 \geq 0$, ${\mathbf{s}^\dagger}_2 < M+||\Lambda||$, ${\mathbf{t}^\dagger}_1 < M+||\Lambda||$, and ${\mathbf{t}^\dagger}_2 \geq 6(K+1)(M+1)||\Lambda||^4 - ||\Lambda||$.

If $\langle 0,1\rangle$ is in the cone of $\mathbf{C}$, we are done by Lemma 6 applied to $\pi^\dagger$ from $\mathbf{s}^\dagger$, which visits only points in $(\mathbb{N}_{\geq M})^2$. Note that all points of π after $\mathbf{s}$ have a second coordinate of at least M. Therefore, the shortening due to Lemma 6 can also be applied to π and result in an admissible path from $\mathbf{s}$.

If $\langle 0,1\rangle$ is not in the cone of $\mathbf{C}\setminus\{\mathbf{0}\}$ then Lemma 4 provides a vector $\mathbf{v}\in\mathbb{Z}^2$ such that $||\mathbf{v}||\leq||\Lambda||$, $\mathbf{v}_{\circlearrowleft}\cdot\langle 0,1\rangle<0$, $\mathbf{v}_{\circlearrowleft}\cdot\mathbf{a}\geq 0$ for all $\mathbf{a}\in\mathbf{C}$, and such that $\mathbf{v}_2>0$ implies $\mathbf{v}\in\mathbf{C}$. Hence, $\mathbf{v}_1<0$ and Lemma 5 (ii) gives us

$$\begin{aligned}\mathbf{v}_{\circlearrowleft}\cdot(\mathbf{t}^\dagger-\mathbf{s}^\dagger)&\geq-(2KN||\Lambda||^2+1)(2||\Lambda||^2)\\&\geq-2K(2N+1)||\Lambda||^4.\end{aligned}\tag{3}$$

But then $\mathbf{v}_2>0$, since the contrary would contradict Equation 3:

$$\begin{aligned}\mathbf{v}_{\circlearrowleft}\cdot(\mathbf{t}^\dagger-\mathbf{s}^\dagger)&<-\mathbf{v}_2(M+||\Lambda||)\\&\quad+\mathbf{v}_1(6(K+1)(M+1)||\Lambda||^4-M-2||\Lambda||)\\&\leq||\Lambda||(M+||\Lambda||)\\&\quad-(6(K+1)(M+1)||\Lambda||^4-M-2||\Lambda||)\\&\leq(-6K(M+1)-6(M+1)+(3+2M))||\Lambda||^4\\&\leq-6K(M+1)||\Lambda||^4\\&\leq-2K(2N+1)||\Lambda||^4,\end{aligned}$$

where the last step follows since $M\geq 6||\Lambda||^3N$. Hence $\mathbf{v}\in\mathbf{C}$.

Recalling $(\mathbf{t}^\dagger-\mathbf{s}^\dagger)_1\geq-\mathbf{s}^\dagger{}_1\geq-\mathbf{s}_1-||\Lambda||$ and

$$\begin{aligned}(\mathbf{t}^\dagger-\mathbf{s}^\dagger)_2&\geq(\mathbf{t}'-\mathbf{s})_2-2||\Lambda||\\&\geq\mathbf{t}_2-(\mathbf{t}-\mathbf{t}')_2-M-2||\Lambda||\\&\geq\mathbf{t}_2-6(K+1)(M+1)||\Lambda||^4-M-2||\Lambda||\\&\geq\mathbf{t}_2-6(K+2)(M+1)||\Lambda||^4,\end{aligned}$$

we then conclude that

$$\begin{aligned}&\mathbf{v}_{\circlearrowleft}\cdot\langle\mathbf{s}_1,-\mathbf{t}_2\rangle\\&\leq\langle-\mathbf{v}_2,\mathbf{v}_1\rangle\cdot(\mathbf{s}^\dagger-\mathbf{t}^\dagger-\langle||\Lambda||,6(K+2)(M+1)||\Lambda||^4\rangle)\\&<2K(2N+1)||\Lambda||^4+\mathbf{v}_2||\Lambda||-\mathbf{v}_1 6(K+2)(M+1)||\Lambda||^4\\&\leq 7(K+2)(M+1)||\Lambda||^5.\end{aligned}$$

□

Roughly speaking, our final case deals with a scenario in which the path consists of two parts. The first part goes from close to the y-axis to close to the x-axis without being close to the x-axis anywhere in between. In the second part it goes back, from close to the x-axis to close to the y-axis without being close to the y-axis anywhere in between. See Figure 3 on the right.

Theorem 10. *Suppose $N\in\mathbb{N}$, $M\geq 8||\Lambda||^4N$, and a 2-SLPS Λ with $K>0$ cycles has a path $\rho\pi$ consisting of one segment ρ from $\mathbf{r}$ to $\mathbf{s}$ and a second segment π from $\mathbf{s}$ to $\mathbf{t}$ such that*

- $\mathbf{r}_1<M$, $\mathbf{r}_2\geq 0$,
- $\mathbf{s}_1\geq 0$, $\mathbf{s}_2<M$,
- $\mathbf{t}_1<M$, $\mathbf{t}_2\geq 19(K+2)(M+1)||\Lambda||^6$, $\mathbf{t}_2\geq\mathbf{r}_2$,
- *all points visited by ρ after $\mathbf{r}$ are in $\mathbb{N}_{\geq M}\times\mathbb{N}$ and*
- *all points visited by π after $\mathbf{s}$ are in $\mathbb{N}\times\mathbb{N}_{\geq M}$.*

There exists $\gamma\in\{0,\dots,2||\Lambda||^3\}$ such that, for all $n\in\{1,\dots,N\}$, $\rho\pi$ has a shortening by $\langle 0,n\gamma\rangle$ which is admissible from $\mathbf{r}$.

Proof. If case (i) of Theorem 9 applies to π from $\mathbf{s}$ then we are done immediately, so assume case (ii) applies to it.

Hence, for some cycle $\mathbf{v}\in\mathbb{Z}_{<0}\times\mathbb{Z}_{>0}$ which occurs in π at least $2||\Lambda||^2N$ times, we have

$$\mathbf{v}_{\circlearrowleft}\cdot\langle\mathbf{s}_1,-\mathbf{t}_2\rangle\;<\;7(K+2)(M+1)||\Lambda||^5.$$

This also implies $\mathbf{s}_1\geq 12(K+1)(M+1)||\Lambda||^4$, since otherwise

$$\begin{aligned}\mathbf{v}_{\circlearrowleft}\cdot\langle\mathbf{s}_1,-\mathbf{t}_2\rangle&>-\mathbf{v}_2 12(K+1)(M+1)||\Lambda||^4-\mathbf{v}_1\mathbf{t}_2\\&\geq-12(K+1)(M+1)||\Lambda||^5+\mathbf{t}_2\\&\geq 7(K+2)(M+1)||\Lambda||^5.\end{aligned}$$

Consequently, Theorem 9 with $N\|\Lambda\|$ for N and with the axes swapped applies to ρ from $\mathbf{r}$.

Suppose that case (ii) of Theorem 9 holds. That is, for some cycle $\mathbf{w}\in\mathbb{Z}_{>0}\times\mathbb{Z}_{<0}$ which occurs in ρ at least $2||\Lambda||^3N$ times, we have

$$\langle-\mathbf{w}_1,\mathbf{w}_2\rangle\cdot\langle\mathbf{r}_2,-\mathbf{s}_1\rangle\;<\;7(K+2)(M+1)||\Lambda||^5.$$

We will reduce the occurrence of cycle $\mathbf{w}$ in ρ by $-\mathbf{v}_1\cdot n$ resulting in a shortening by $-\mathbf{v}_1\cdot n\cdot\mathbf{w}$.

If case (i) of Theorem 9 with $N||\Lambda||$ for N and with the axes swapped applies to ρ from $\mathbf{r}$, there is a value $\gamma'\in\{1,\dots,2||\Lambda||^2\}$ such that we can shorten ρ by $-\mathbf{v}_1\cdot n\cdot\langle\gamma',0\rangle$. For convenience, we define $\mathbf{w}\overset{\text{def}}{=}\langle\gamma',0\rangle$ in this case.

Either way, the resulting shortened version of ρ is admissible from $\mathbf{r}$. In both cases, the second coordinate of points cannot decrease due to the shortening (note that $\mathbf{w}_2\leq 0$). The first coordinate may decrease but by at most $||\Lambda||\cdot N||\Lambda||\cdot 2||\Lambda||^2=2N||\Lambda||^4<M$. Therefore, the shortened version of ρ is still admissible since, prior to the shortening, all points visited by ρ after $\mathbf{r}$ have a first coordinate of at least M.

Note that, while ρ is still admissible after the shortening, $\rho\pi$ may not be admissible anymore. Therefore, we also need to shorten π appropriately to counter the effect that the shortening of ρ may have had on the first coordinate. We shorten π by reducing the number of occurrences of cycle $\mathbf{v}$ by $\mathbf{w}_1\cdot n$. We now argue that such a shortened version of π is admissible from $\mathbf{s}+\mathbf{v}_1\cdot n\cdot\mathbf{w}$.

Following Theorem 9, π consists of two parts: a prefix of π, π' for which all intermediate points lie in $(\mathbb{N}_{\geq M})^2$, and the remaining path after π'. Note that the cycle $\mathbf{v}$ is part of the path π'. Therefore the target point of π' as well as all points on the second part of π experience an increase of their first coordinate by $-\mathbf{v}_1\cdot n\cdot\mathbf{w}_1$. Hence, after the shortening, all points on π starting at $\mathbf{s}$ have a first coordinate of at least $\min\{M,-\mathbf{v}_1\cdot n\cdot\mathbf{w}_1\}=-\mathbf{v}_1\cdot n\cdot\mathbf{w}_1$. Reducing the repetitions of the cycle $\mathbf{v}$ by $\mathbf{w}_1\cdot n$ can decrease the second coordinates of points on the path by no more than $\mathbf{w}_1\cdot n\cdot\mathbf{v}_2\leq 2||\Lambda||^4N<M$ but all points visited by π prior to the shortening lie in $\mathbb{N}\times\mathbb{N}_{\geq M}$. Altogether we conclude that the shortening of π is not only admissible from $\mathbf{s}$, but even admissible from $\mathbf{s}+\mathbf{v}_1\cdot n\cdot\mathbf{w}$.

Overall, we have a shortened version of ρ going from $\mathbf{r}$ to $\mathbf{s}+\mathbf{v}_1\cdot n\cdot\mathbf{w}$ that is admissible. This is followed by a shortened version of π going from $\mathbf{s}+\mathbf{v}_1\cdot n\cdot\mathbf{w}$ to $\mathbf{t}+\mathbf{v}_1\cdot n\cdot\mathbf{w}-\mathbf{w}_1\cdot n\cdot\mathbf{v}=\mathbf{t}-n\cdot\langle 0,\mathbf{w}_1\mathbf{v}_2-\mathbf{w}_2\mathbf{v}_1\rangle$ and which is admissible as well.

Since we successfully shortened $\rho\pi$ by $n\cdot\langle 0,\mathbf{w}_1\mathbf{v}_2-\mathbf{w}_2\mathbf{v}_1\rangle$ it only remains to show that $\mathbf{w}_1\mathbf{v}_2-\mathbf{w}_2\mathbf{v}_1\in\{0,\dots,2||\Lambda||^3\}$. Clearly, $\mathbf{w}_1\mathbf{v}_2-\mathbf{w}_2\mathbf{v}_1<\mathbf{w}_1\mathbf{v}_2\leq 2||\Lambda||^3$. On the other hand, it cannot be that $\mathbf{v}_1\mathbf{w}_2>\mathbf{v}_2\mathbf{w}_1$, because it implies

$$\begin{aligned}\mathbf{t}_2&\leq\langle\mathbf{v}_2\mathbf{w}_1,\mathbf{v}_1\mathbf{w}_2\rangle\cdot\langle-\mathbf{r}_2,\mathbf{t}_2\rangle\\&=-\mathbf{r}_2\mathbf{v}_2\mathbf{w}_1+\mathbf{t}_2\mathbf{v}_1\mathbf{w}_2\\&=\mathbf{v}_2\cdot\langle-\mathbf{w}_1,\mathbf{w}_2\rangle\cdot\langle\mathbf{r}_2,-\mathbf{s}_1\rangle-\mathbf{w}_2\cdot\langle-\mathbf{v}_2,\mathbf{v}_1\rangle\cdot\langle\mathbf{s}_1,-\mathbf{t}_2\rangle\\&<(\mathbf{v}_2-\mathbf{w}_2)\cdot 7(K+2)(M+1)||\Lambda||^5\\&\leq 14(K+2)(M+1)||\Lambda||^6.\end{aligned}$$

□

Our penultimate theorem states that it is not possible for a shortest reachability witness to visit a point $\mathbf{f}$ whose norm is much larger than the norms of the last point close to the axes before visiting $\mathbf{f}$ and the first point close to the axes after visiting $\mathbf{f}$.

Theorem 11. *Suppose a* 2*-SLPS* Λ *with* K *cycles and with* $||\Lambda|| > 0$ *has a path* π *from point* $\mathbf{s}$ *to point* $\mathbf{t}$ *such that*

- *all points visited by* π *from* $\mathbf{s}$ *are in* $(\mathbb{N}_{\geq 6||\Lambda||^3})^2$ *and*
- *some point* $\mathbf{f}$ *visited by* π *from* $\mathbf{s}$ *satisfies*

$$\|\mathbf{f}\| \;>\; 3\|\Lambda\|^2 \cdot \|\{\mathbf{s},\mathbf{t}\}\| + 7.5\|\Lambda\|^5 K.$$

There is a shortening of π *by* $\mathbf{0}$ *that is admissible from* $\mathbf{s}$.

Proof. We have that

$$\begin{aligned}|\pi| &\geq 2(\|\mathbf{f}\| - \|\{\mathbf{s},\mathbf{t}\}\|)/\|\Lambda\| \\ &> 4\|\Lambda\| \cdot \|\{\mathbf{s},\mathbf{t}\}\| + 15\|\Lambda\|^4 K \\ &\geq 4\|\Lambda\|\,\|\mathbf{t}-\mathbf{s}\| + (K2\|\Lambda\|^2+1)(4\|\Lambda\|^2+1).\end{aligned}$$

In particular, $\mathbf{C} = \mathrm{Cycles}_{\geq 2\|\Lambda\|^2}(\Lambda,\pi)$ cannot be empty. Suppose the cone of $\mathbf{C}$ does not contain $\mathbf{0}$. Then Lemma 3 provides a vector $\mathbf{p}$ with $\|\mathbf{p}\| \leq 2\|\Lambda\|$ and $\mathbf{p}\cdot\mathbf{c} > 0$ for all $\mathbf{c} \in \mathbf{C}$. By Lemma 5 (i) we then get

$$\begin{aligned}4\|\Lambda\|\,\|\mathbf{t}-\mathbf{s}\| &\geq \mathbf{p}\cdot(\mathbf{t}-\mathbf{s}) \\ &\geq |\pi| - (K2\|\Lambda\|^2+1)(4\|\Lambda\|^2+1),\end{aligned}$$

which contradicts the inequation above. So the cone of $\mathbf{C}$ contains $\mathbf{0}$ and we finish by Lemma 6 with $N = 1$ and $\mathbf{c} = \mathbf{0}$. ☐

We are now equipped to establish that 2-dimensional simple linear path schemes have pseudo-polynomially bounded reachability witnesses:

Theorem 12. *Suppose* Λ *is a* 2*-SLPS with* K *cycles. For any shortest admissible path from* $\mathbf{0}$ *to* $\mathbf{0}$*, the norms of all points visited are at most* $2914.5K\|\Lambda\|^{15}$.

Proof. We can assume $K, \|\Lambda\| > 0$. Consider any shortest admissible $\pi \in \Lambda$ from $\mathbf{0}$ to $\mathbf{0}$, and let $M \stackrel{\text{def}}{=} 16\|\Lambda\|^7$.

First, we show that at all points visited by π where one coordinate is less than M, the other coordinate must be less than

$$\begin{aligned}M' \;\stackrel{\text{def}}{=}\; 969K\|\Lambda\|^{13} &= 19(3K)(17\|\Lambda\|^7)\|\Lambda\|^6 \\ &\geq 19(K+2)(M+1)\|\Lambda\|^6.\end{aligned}$$

To see this, assume the contrary and let $\mathbf{t} \in \mathbb{N}^2$ be a point visited by π from $\mathbf{0}$ such that, w.l.o.g., $\mathbf{t}_1 < M$ and $\mathbf{t}_2 \geq M'$. Further assume that $\mathbf{t}$ is a point with maximum $\mathbf{t}_2$ among all points with this property. Then we can extract a subpath ρ by following π backwards, starting in $\mathbf{t}$ until for the first time a point $\mathbf{s}$ is visited that satisfies $\mathbf{s}_2 < M$ and then further, until for the first time a point $\mathbf{r}$ is visited with $\mathbf{r}_1 < M$. (Here it may be the case that $\mathbf{s}$ and $\mathbf{r}$ are the same point, i.e. the latter path segment is empty.) On this path Theorem 10 is applicable with $N = 2\|\Lambda\|^3$. So there exist $\gamma \in \{0,\ldots 2\|\Lambda\|^3\}$ and shortenings by $\langle 0, n\gamma\rangle$ for all $n \in \{1,\ldots N\}$, admissible from the point $\mathbf{r}$. If $\gamma = 0$ then this directly contradicts the minimality of π. Otherwise we can, analogously, extract a subpath ρ' by following π forwards from $\mathbf{t}$ to some $\mathbf{r}'$ and then reversing, so that Theorem 10 provides $\gamma' \in \{0,\ldots 2\|\Lambda\|^3\}$ and shortenings by $\langle 0, n\gamma'\rangle$ for all $n \in \{1,\ldots N\}$, admissible backwards from $\mathbf{r}'$. See Figure 4 for an illustration. Together, this means there is a shortening of π by $\mathbf{0}$; a contradiction with the minimality assumption.

To show the claim of the theorem, assume that π visits some point $\mathbf{f}$ whose norm exceeds $2914.5K\|\Lambda\|^{15} \geq 3\|\Lambda\|^2 M' + 7.5\|\Lambda\|^5 K$. Then we can partition $\mathbf{0} \xrightarrow{\pi} \mathbf{0}$ as $\mathbf{0} \xrightarrow{\rho} \mathbf{s} \xrightarrow{\sigma} \mathbf{f} \xrightarrow{\sigma'} \mathbf{t} \xrightarrow{\tau} \mathbf{0}$ where $\|\mathbf{s}\|, \|\mathbf{t}\| < M'$ and all other points visited by $\sigma\sigma'$ from $\mathbf{s}$ are in $(\mathbb{N}_{\geq M})^2$. But then Theorem 11 provides a shortening of $\sigma\sigma'$ that is admissible from $\mathbf{s}$, and thus a shortening of π admissible from $\mathbf{0}$, again contradicting the minimality assumption. ☐

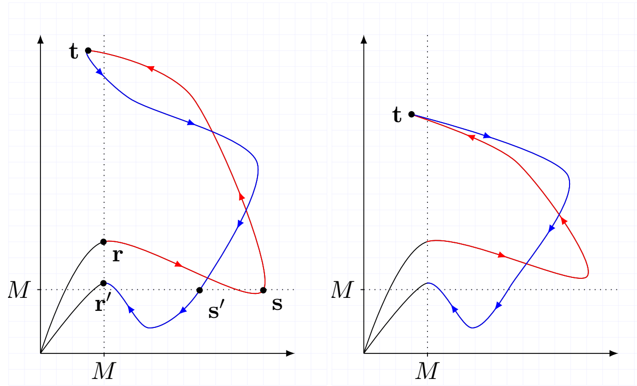

Figure 4. In Theorem 12, we identify two path segments, the red one from $\mathbf{r}$ to $\mathbf{t}$ via $\mathbf{s}$ and the blue one from $\mathbf{t}$ to $\mathbf{r}'$ via $\mathbf{s}'$ pictured on the left. Both can be shortened via Theorem 10 (where the theorem is applied to the revers of the blue path). Both shortenings combined result in a new, shorter path from $\mathbf{0}$ to $\mathbf{0}$. In the new path, the point corresponding to $\mathbf{t}$ has moved down. This is pictured on the right.

5. Finish: 2-VASS

Blondin et al. [1, Thm. 1] showed that 2-VASS can be flattened, i.e., their reachability relation can be expressed by a finite set of polynomially bounded linear path schemes:

Theorem 13. *For every* 2*-VASS* V *with* n *states over an alphabet* $\mathbf{A} \subseteq \mathbb{Z}^2$*, there exist finitely many LPSs* $\Lambda_1, \Lambda_2, \ldots, \Lambda_k \subseteq V$ *such that* $\xrightarrow{V} = \bigcup_{i=1}^{k} \xrightarrow{\Lambda_i}$ *and* $|\Lambda_i| \leq (\|\mathbf{A}\| + n)^{O(1)}$ *for all* $1 \leq i \leq k$.

Small witness theorems for 2-dimensional LPSs therefore carry over to 2-VASS. To apply our small witness theorem for *simple* LPSs a further reduction (Theorem 15 below) is necessary. We will use the following fact.

Lemma 14. *Suppose* $\mathbf{A} \subseteq^{\mathit{fin}} \mathbb{Z}^d$*,* $\pi \in \mathbf{A}^*$*,* $m \in \mathbb{N}$ *and* $\mathbf{s} \in \mathbb{N}^d$*. Then* π^{m+2} *is admissible from* $\mathbf{s}$ *if and only if* $\pi(\Sigma\pi)^m\pi$ *is admissible from* $\mathbf{s}$.

Proof. The 'only if' direction is immediate; for the other direction observe that if $\pi(\Sigma\pi)^m\pi$ is admissible from $\mathbf{s}$, then there is $\mathbf{t} \in \mathbb{N}^2$ such that $\mathbf{s} \xrightarrow{\pi(\Sigma\pi)^m} \mathbf{t}$ and π is admissible both from $\mathbf{s}$ and $\mathbf{t}$. Let $\mathbf{g} \in \mathbb{N}^d$ be minimal such that π is admissible from it. We show that π is admissible from all points $\mathbf{s} + \Sigma\pi \cdot i$ for $0 \leq i \leq m$. Suppose this fails for some i and $\mathbf{v} \stackrel{\text{def}}{=} (\mathbf{s} + \Sigma\pi \cdot i) \ngeq \mathbf{g}$. Then $\mathbf{v}_j < \mathbf{g}_j$ for some dimension $1 \leq j \leq d$. Since $\mathbf{s} \geq \mathbf{g}$, it must hold that $(\Sigma\pi)_j < 0$ and because $\mathbf{t} = \mathbf{s} + \Sigma\pi \cdot (m+1)$, also $\mathbf{t}_j < \mathbf{g}_j$ and consequently, $\mathbf{t} \ngeq \mathbf{g}$. Contradiction. ☐

Theorem 15. *For every LPS* L *there are finitely many SLPSs* $\Lambda_1, \Lambda_2, \ldots, \Lambda_k$ *such that* $\xrightarrow{L} = \bigcup_{i=1}^{k} \xrightarrow{\Lambda_i}$ *and*

1. *For all* $i \leq k$*,* $|\Lambda_i| \leq 4|L|$ *and* $\|\Lambda_i\| \leq 2\|L\| \cdot |L|$.
2. *For every path* $\pi \in \Lambda_i$ *there exists* $\pi' \in L$ *with* $|\pi'| \leq |\pi| \cdot |L|$ *and* $\xrightarrow{\pi} \subseteq \xrightarrow{\pi'}$.

Proof. The idea is first to split L into a finite set S of LPS such that each of them predetermines, for each cycle, if it can be used zero, one or more than one times. Clearly, $\bigcup S = L$ and the maximum length of any resulting LPS is $3|L|$. In each such LPS Λ we then replace occurrences of subexpressions $\beta_i\beta_i^*\beta_i$ by subexpressions $\beta_i(\Sigma\beta_i)^*\beta_i$, which does not increase the length and can only increase the norm to $\|\Lambda\| \leq \|L\| \cdot |L|$. By Lemma 14 this moreover does not change the relation $\xrightarrow{\Lambda}$ and guarantees the second claimed property. It remains to introduce a total of at most $|L|$ many cycles $\mathbf{0}^*$ into the unstarred segments to make the LPS simple. $\square$

Theorem 16. *2-VASS have pseudo-polynomially long reachability witnesses.*

Proof. Suppose V is a 2-VASS with n states over an alphabet $\mathbf{A} \subseteq \mathbb{Z}^2$ and $\mathbf{s}, \mathbf{t} \in \mathbb{N}^2$ are such that $\mathbf{s} \xrightarrow{\pi} \mathbf{t}$ for some path $\pi \in V$.

First note that a 2-VASS $V' \stackrel{\text{def}}{=} (\mathbf{s})\, V\, (-\mathbf{t})$, obtained from V by adding two states, has an admissible path $\pi' = (\mathbf{s})\pi(-\mathbf{t})$ from $\mathbf{0}$ to $\mathbf{0}$. By Theorems 13 and 15 there is an 2-SLPS Λ such that:

- $|\Lambda|$ and $\|\Lambda\|$ are polynomial in n and $\|\mathbf{A} \cup \{\mathbf{s}, \mathbf{t}\}\|$;
- Λ has an admissible path from $\mathbf{0}$ to $\mathbf{0}$;
- for every path $\rho \in \Lambda$, there exists $\rho' \in V'$ with $\xrightarrow{\rho} \subseteq \xrightarrow{\rho'}$ and with $|\rho'|$ polynomial in $|\rho|$, n and $\|\mathbf{A} \cup \{\mathbf{s}, \mathbf{t}\}\|$.

Now, by Theorem 12, we have $\mathbf{0} \xrightarrow{\rho} \mathbf{0}$ for some path $\rho \in \Lambda$ with $|\rho|$ polynomial in $|\Lambda|$ and $\|\Lambda\|$, and thus polynomial in n and $\|\mathbf{A} \cup \{\mathbf{s}, \mathbf{t}\}\|$. Hence there exists $\rho' \in V'$ such that $\mathbf{0} \xrightarrow{\rho'} \mathbf{0}$ and $|\rho'|$ is polynomial in n and $\|\mathbf{A} \cup \{\mathbf{s}, \mathbf{t}\}\|$, as required. $\square$

A direct consequence is that a nondeterministic algorithm that guesses a bounded witness on the fly requires space logarithmic in the number of states and the infinity norms of action, source and target vectors. Recall also that already 0-VASS are essentially directed graphs.

Corollary 17. *The reachability problem for 2-VASS with integers given in unary is NL-complete.*

6. Conclusion

That the covering and boundedness problems for VASS given in unary are NL-complete for any *fixed* dimension has been known for thirty years [7]. This contribution suggests that, possibly, the same is true for the reachability problem.

If that is too challenging, how about restricting to flat VASS, i.e. linear path schemes, and attempting to extend the machinery developed here to dimension 3 in order to close the gap between NL hardness and NP membership [1] in that case?

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References

[1] M. Blondin, A. Finkel, S. Göller, C. Haase, and P. McKenzie. Reachability in two-dimensional vector addition systems with states is PSPACE-complete. In *LICS*, pages 32–43. IEEE, 2015. doi: 10.1109/LICS.2015.14.

[2] E. Cardoza, R. J. Lipton, and A. R. Meyer. Exponential space complete problems for Petri nets and commutative semigroups: Preliminary report. In *STOC*, pages 50–54. ACM, 1976. doi: 10.1145/800113.803630.

[3] S. M. German and A. P. Sistla. Reasoning about systems with many processes. *J. ACM*, 39(3):675–735, 1992. doi: 10.1145/146637.146681.

[4] C. Haase, S. Kreutzer, J. Ouaknine, and J. Worrell. Reachability in succinct and parametric one-counter automata. In *CONCUR*, volume 5710 of *LNCS*, pages 369–383. Springer, 2009. doi: 10.1007/978-3-642-04081-8_25.

[5] J. Leroux and S. Schmitz. Demystifying reachability in vector addition systems. In *LICS*, pages 56–67. IEEE, 2015. doi: 10.1109/LICS.2015.16.

[6] R. J. Lipton. The reachability problem requires exponential space. Technical Report 62, Yale University, 1976. URL `http://cpsc.yale.edu/sites/default/files/files/tr63.pdf`.

[7] L. E. Rosier and H. Yen. A multiparameter analysis of the boundedness problem for vector addition systems. *J. Comput. Syst. Sci.*, 32(1):105–135, 1986. doi: 10.1016/0022-0000(86)90006-1.

[8] S. Schmitz. Automata column: The complexity of reachability in vector addition systems. *ACM SIGLOG News*, 3(1), 2016. URL `http://siglog.hosting.acm.org/wp-content/uploads/2016/01/siglog_news_7.pdf`.

[9] L. G. Valiant and M. S. Paterson. Deterministic one-counter automata. *J. Comput. Syst. Sci.*, 10(3):340–350, 1975. doi: 10.1016/S0022-0000(75)80005-5.