Reachability in Two-dimensional Unary Vector Addition Systems with States is NL-complete

Matthias Englert Ranko Lazić **Patrick Totzke**

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Vector Addition Systems

Definition

A d-dimensional VAS is a finite set of vectors $A \subseteq \mathbb{Z}^d$.
For $\mathbf{v}, \mathbf{v}' : \mathbb{N}^d$ it has a step $\mathbf{v} \xrightarrow{\mathbf{a}} \mathbf{v}'$ if $\mathbf{v}' = \mathbf{v} + \mathbf{a}$.

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Let's lift this to words over A:

$$\xrightarrow{\varepsilon} \stackrel{\text{def}}{=} Id_{\mathbb{N}^d} \qquad \xrightarrow{aw} \stackrel{\text{def}}{=} \xrightarrow{w} \circ \xrightarrow{a}$$

where ε is the empty word, $a \in A$, $w \in A^*$.

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The (VASS) Reachability Problem

Input: a regular language $L \subseteq A^*$ over $A \subseteq \mathbb{Z}^d$

Question: does $\mathbf{0} \xrightarrow{L} \mathbf{0}$ hold?

Reachability in Two-Dimensional Vector Addition Systems with States is PSPACE-complete

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Abstract—**Known to be decidable since 1981, there still remains a huge gap between the best known lower and upper bounds for the reachability problem for vector addition systems with states (VASS). Here the problem is shown PSPACE-complete in the two-dimensional case, vastly improving on the doubly exponential time bound established in 1986 by Howell, Rosier, Huynh and Yen. Coverability and boundedness for two-dimensional VASS are also shown PSPACE-complete, and reachability in two-dimensional VASS and in integer VASS under unary encoding are considered.**

I. Introduction

Petri nets have a long history. Since their introduction [19] by Petri in 1962, thousands of papers on Petri nets have been published. Nowadays, Petri nets find a variety of applications, ranging, for instance, from modeling of biological, chemical and business processes to the formal verification of concurrent programs, see *e.g.* [1], [4], [8], [21], [27]. For the analysis of their algorithmic properties, Petri nets are often equivalently viewed as *vector addition systems with states (VASS)*, and we will adopt this view throughout this paper. A VASS comprises a finite-state controller with a finite number of counters ranging over the natural numbers. The number of counters is usually referred to as the *dimension* of the VASS, and we write d-VASS to denote VASS in dimension d. When taking a transition, a VASS can add or subtract an integer from a counter, provided that the resulting counter values are greater than or equal to zero; otherwise the transition is blocked. A configuration of a VASS is a tuple consisting of a control state and an assignment of natural numbers to the counters. The central decision problem for VASS is *reachability*: given two configurations, is there a path connecting them in the infinite

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Milestones in the work on the complexity of the VASS reachability problem include Lipton's 1976 proof that the problem, regardless of the choice of encoding for numbers but without fixed dimension, is EXPSPACE-hard [17]. Yet our knowledge of the situation for any fixed dimension d is vastly lacking. For 1-VASS, reachability under unary encoding is easily seen to be NL-complete: the hardness is inherited from graph reachability and the upper bound follows from a simple pumping argument. Under binary encoding, 1-VASS reachability is known to be NP-complete [5]. As a substantial contribution towards showing decidability of the general problem, Hopcroft and Pansiot in 1979 showed the two-dimensional case decidable [9]. At the core of their proof lies an intricate algorithm that implicitly exploits the fact that the reachability set of a 2-VASS is semi-linear. Exhibiting a 3-VASS with a reachability set that is not semi-linear, Hopcroft and Pansiot could show that their method breaks down for d-VASS for any d greater than 2. Further complexity aspects were left unanswered in [9]. In 1986, Howell, Rosier, Huynh and Yen [10] observed that Hopcroft and Pansiot's algorithm runs in nondeterministic doubly-exponential time, under both unary and binary encoding. They then managed to improve this bound from nondeterministic to deterministic

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and between NL and NP if numbers are given in unary

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Our Contribution

Theorem

For regular $L \subseteq (\mathbb{Z} \times \mathbb{Z})^*$, shortest paths witnessing $\mathbf{0} \xrightarrow{L} \mathbf{0}$ are bounded by $B \stackrel{def}{=} (|L| + \|L\|)^{\mathcal{O}(1)}$.

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- on-the-fly guessing a witness uses $\log(B)$ space.
- implies the PSPACE upper bound of Blondin et al.'15
- closes the gap to NL completeness for unary case.

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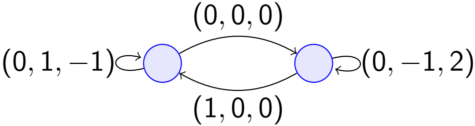

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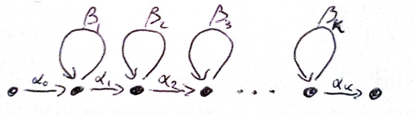

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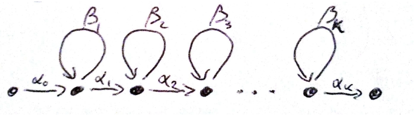

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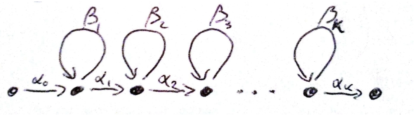

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Leroux and Sutre '04
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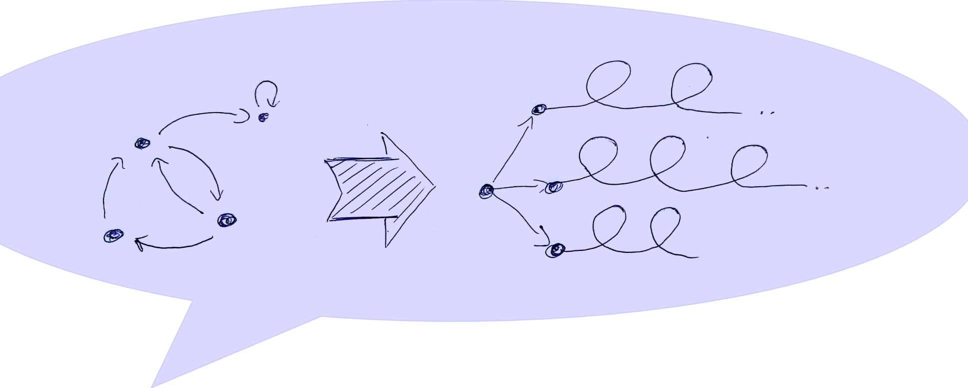
2-dim. VASS are flattable



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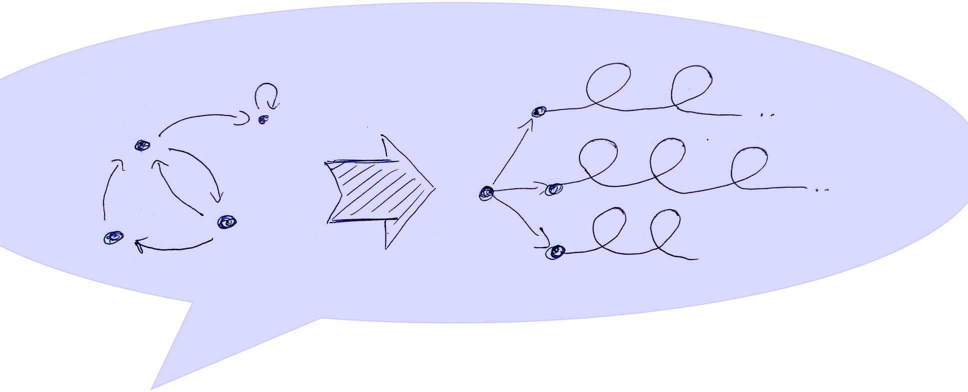

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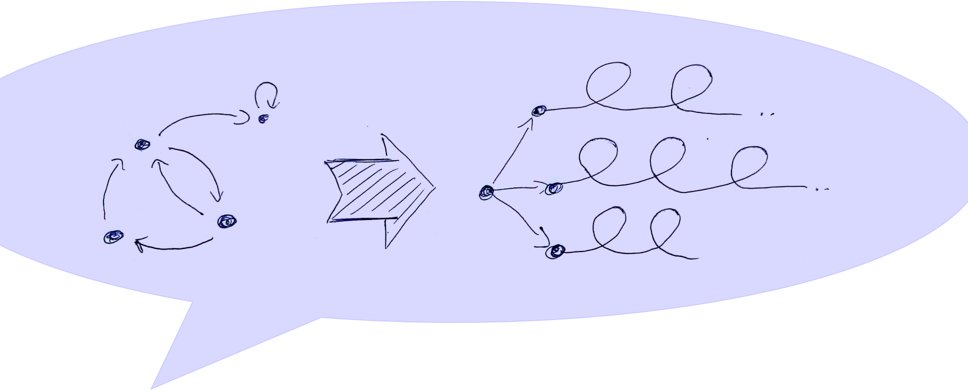
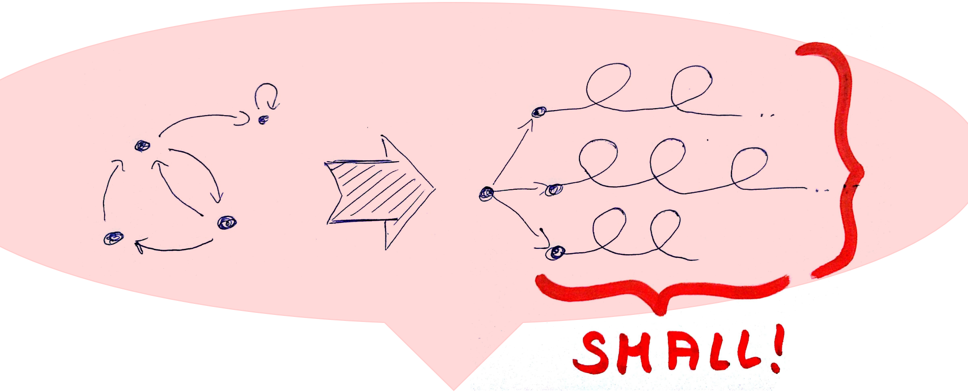


2-dim. VASS are polynomially flattable



Blondin et al. '15

"Reachability in 2-VASS Is PSPACE-Complete"

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Our Construction

Given a linear path scheme

$$\Lambda = \alpha_0 \beta_1^* \alpha_1 \beta_2^* \ldots \beta_K^* \alpha_K$$

we say $\pi \stackrel{\text{def}}{=} \mathbf{v_1 v_2 \ldots v_k} \in \Lambda$ *visits* $\mathbf{p} \in \mathbb{Z}^2$ if $\mathbf{p} = \mathbf{v_1} + \mathbf{v_2} + \cdots + \mathbf{v_j}$ for some $j \leq k$.

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Rational Cones

Definition

The *cone* spanned by $A \subseteq \mathbb{Z}^2$ is the smallest set satisfying

- $Cone(A) \supseteq A$
- $Cone(A) = Cone(A) + Cone(A)$
- $Cone(A) = Cone(A) \cdot \mathbb{Q}_{>0}$.

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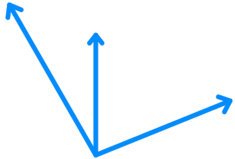
The *cone* spanned by $A \subseteq \mathbb{Z}^2$ is

$$Cone(A) \stackrel{\text{def}}{=} \left\{ \sum_{i=1}^{l} a_i \mathbf{v_i} \mid a_i \in \mathbb{Q}_{>0},\ \mathbf{v_i} \in A,\ l > 0 \right\}$$

Rational Cones



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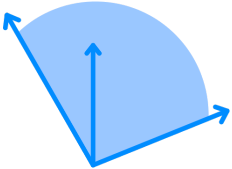

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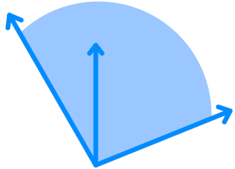

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If a cone is not contained in a half-plane then it contains $\mathbf{0}$.

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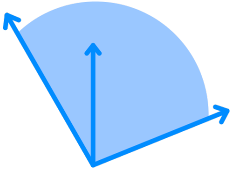

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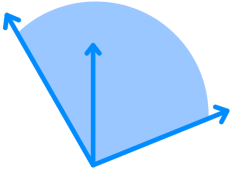

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If $\mathbf{v} \in Cone(A)$ and $\mathbf{w} \in \mathbb{Z}^2$ is such that $\|\mathbf{w}\| \leq \|A\|$ and $\mathbf{v} \cdot \mathbf{w}$ is maximal, then $\mathbf{w} \in Cone(A)$.

Cones and Paths

For $\pi = \alpha_0 \beta_1^{n_1} \alpha_1 \beta_2^{n_2} \dots \beta_K^{n_K} \alpha_K \in \Lambda$ write

$$Cycles_B(\pi) \stackrel{\text{def}}{=} \{\beta_i \mid n_i \geq B\} \subseteq S$$

for those cycles β_i occurring $n_i \geq B$ times and let

$$Cone_B(\pi) \stackrel{\text{def}}{=} Cone(Cycles_B(\pi)).$$

A Cut Lemma

Observation

If $\mathbf{s}\pi$ visits only points in $\mathbb{N}_{\geq B} \times \mathbb{N}_{\geq B}$ and π' is a subword of π such that $B \geq (|\pi| - |\pi'|) \cdot \|S\|$ then $\mathbf{s}\pi'$ visits only points in $\mathbb{N}^2$.

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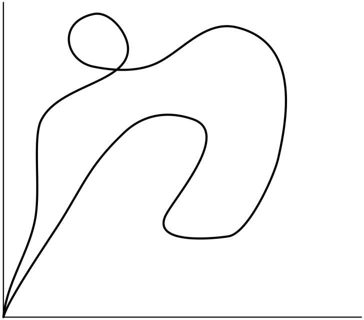

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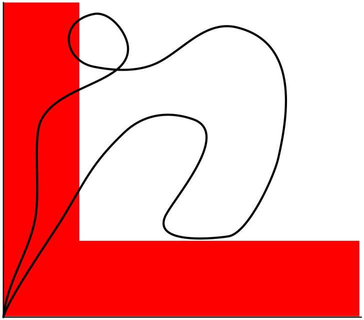

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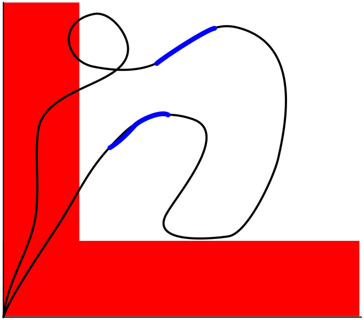

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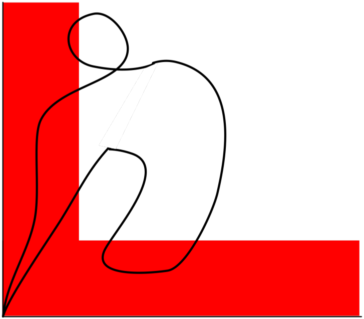

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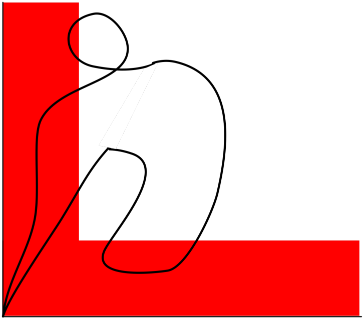
Cut Lemma

If σ is part of a shortest witness and visits only points in $\mathbb{N}^2_{\geq B}$, then $\mathbf{0} \notin Cone_B(\sigma)$.



Existence of small witnesses

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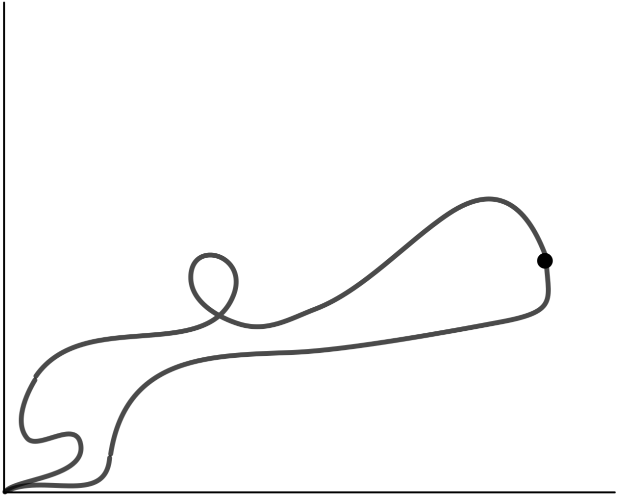

Existence of small witnesses assuming the Magic Lemma™

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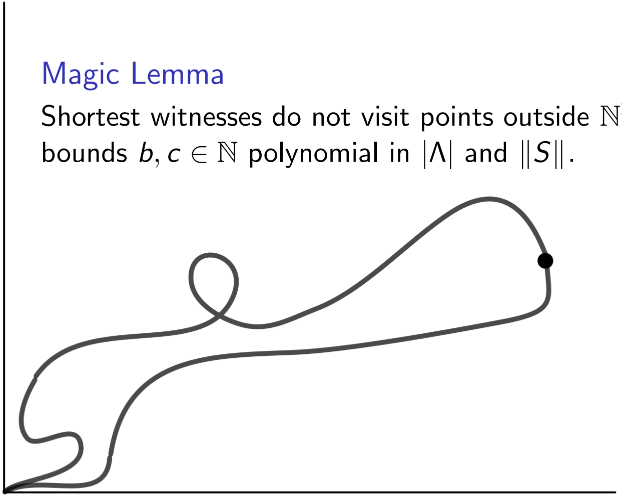
Magic Lemma

Shortest witnesses do not visit points outside $\mathbb{N}^2_{\leq c} \cup \mathbb{N}^2_{\geq b}$ for some bounds $b, c \in \mathbb{N}$ polynomial in $|\Lambda|$ and $\|S\|$.



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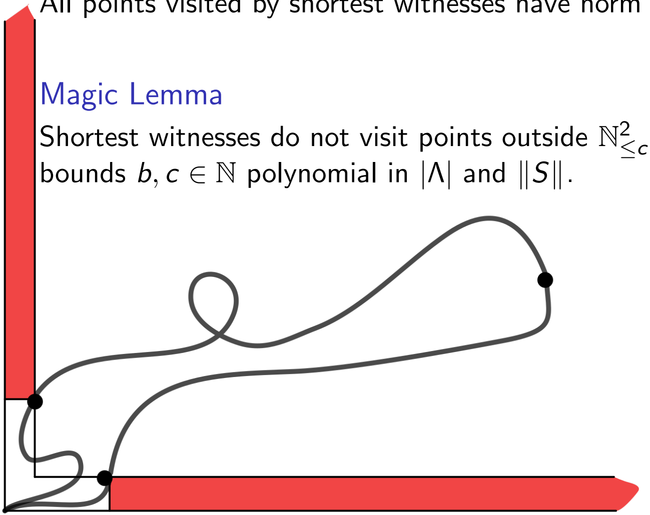

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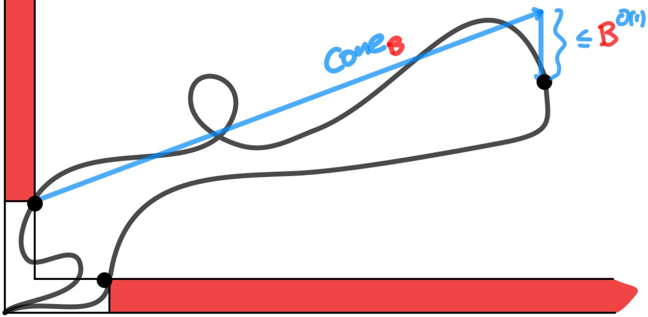

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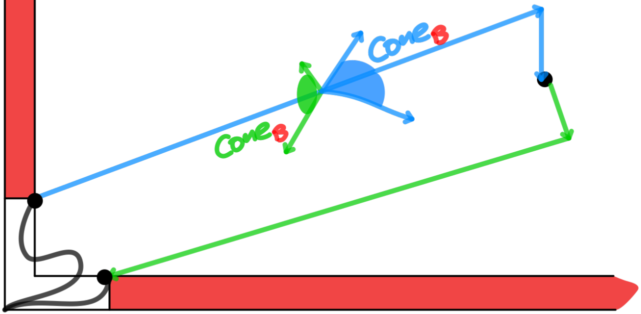

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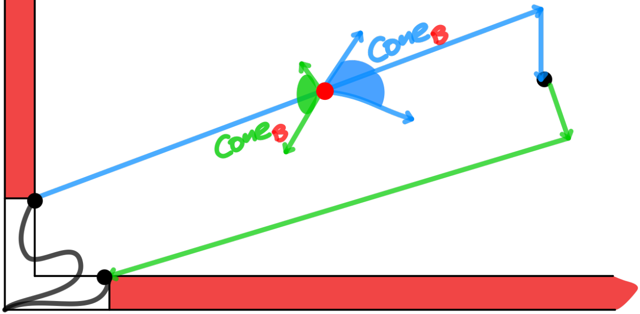

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2-VASS have $(|L| \cdot \|L\|)^{\mathcal{O}(1)}$ long reachability witnesses.

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Outlook

- Does this generalise to $(d > 2)$-dimensional *flattable* VASS?
- regular separability of 1-VASS languages
- 1-dim. *pushdown* VASS, or 2-dim. *branching* VASS?


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danke
謝謝
ngiyabonga
teşekkür ederim
mersi
barka
welalin
tack
paldies
grazzi
thank you
gracia
dank je
misaotra
matondo
faafetai lava
vinaka
blagodaram
maururu
koszonom
dankon
djiere dieuf
moch
дякую
Outlook
ekuje
dekuji
chnorakaloutioun
gratias ago
gràcies
sulpay
go raibh mai
sukriya
kop khun krap
takk
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talko
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didi madloba
najis tuke
ありがとう
terima kasih
tanemirt
grazie
arigatô
rahmet
kam sah hamnida
rahmat
diolch
dhanyavadagalu
shukriya
감사합니다
xiexie
ευχαριστώ
merci