Approximating Weak Bisimilarity of Basic Parallel Processes

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This paper explores the well known approximation approach to decide weak bisimilarity of *Basic Parallel Processes*. We look into how different refinement functions can be used to prove weak bisimilarity decidable for certain subclasses. We also show their limitations for the general case. In particular, we show a lower bound of $\omega * \omega$ for the approximants which allow weak steps and a lower bound of $\omega + \omega$ for the approximants that allow sequences of actions. The former lower bound negatively answers the open question of Jančar and Hirshfeld.

1 Introduction

Basic Parallel Processes (BPP) were introduced by Christensen [1] as derivations of commutative context-free grammars and are equi-expressible with communication-free Petri nets or process algebra using action prefixing, choice and full merge only. We are interested in deciding the problem of *weak bisimilarity* for BPP, which remains unresolved even for normed systems.

Christensen, Hirshfeld and Moller first prove the decidability of strong bisimulation between BPP [2], Srba and Jančar [16, 14] show the PSPACE completeness of the problem. For the subclass of normed systems – where every process has a finite distance to termination – a polynomial time algorithm for bisimulation exists [11]. On the negative side, Hirshfeld [9] proves trace equivalence undecidable for BPP and Hüttel [12, 13] shows that indeed all equivalences that lie between strong bisimulation and trace equivalence in the linear/branching time spectrum [7] are undecidable.

The main obstacle for deciding weak bisimulation is that one abstracts from silent moves and therefore allows for infinite branching. Weak bisimilarity is known to be PSPACE-hard for the whole class [16] and still NP-hard [19] for the subclass of totally normed systems, which forbids variables of zero and infinite norm. Stirling [18] showed that it is decidable for a non-trivial subclass that still allows infinite branching albeit in a restricted form. Branching bisimulation for normed BPP is shown to be decidable in [3]. However, the technique used there cannot be easily transferred to work also for weak bisimulation. The problem is that in weak bisimulation games Duplicator can go through many equivalence classes when making a move. This makes it hard to find a connection between the sizes of Duplicators configurations before and after move.

Milner originally defines (weak) bisimulation by refinement as the limit of a decreasing sequence of approximants. This definition is known to coincide with the more customary co-inductive definition due to Park but the sequence of approximants does not necessarily converge at a finite level for infinitely branching systems.

We explore the *approximation approach* which is outlined as follows. Weak bisimilarity is a congruence over a commutative monoid and therefore semi-linear [4], which means we can enumerate all candidate relations. The fact that the weak bisimulation condition is expressible in Presburger Arithmetic means that we can determine for each such candidate if it is a weak bisimulation that contains a given pair. Hence, a semi-decision procedure for inequivalence immediately implies decidability. The approximation method discussed here yields such a semi-decision procedure under two assumptions: 1) $\approx$ is finitely approximable: The sequence of approximants stabilizes at level ω, the first limit ordinal. 2) Each approximant $\approx_o$ for $o < \omega$ is decidable. If both hold true, one can simply iterate through all approximants

B. Luttik and M.A. Reniers (Eds.): Combined Workshop on Expressiveness in
Concurrency and Structural Operational Semantics (EXPRESS/SOS 2012)
EPTCS 89, 2012, pp. 99–113, doi:10.4204/EPTCS.89.8

and for each one check if the given pair of processes is not contained. The first condition guarantees that this procedure terminates after finitely many rounds for any pair of inequivalent processes.

Because finite approximation fails for most interesting subclasses we focus on more rigorous refinement functions than the ones typically considered. We successfully apply the approximation method to restricted classes of BPP: We derive a decision procedure for checking weak bisimulation for a class defined by Stríbrná in [19] that allows only a single visible action and no variables of 0 norm. Moreover, we provide a new proof for the decidability of weak bisimulation for the class defined by Stirling [18].

We show a lower bound of $\omega * \omega$ for the convergence index of the approximants considered previously, falsifying a conjecture that is attributed to Hirshfeld and Jančar[1] that approximants stabilise at level $\omega+\omega$. Moreover we show that the most powerful notion of approximation under consideration, for which the individual approximants do not even need to be decidable themselves are not guaranteed to converge below level $\omega+\omega$.

2 Preliminaries

We write $V^{\otimes}$ for the set of all multisets over the finite domain V, $\alpha\beta$ for the multiset union of $\alpha,\beta \in V^{\otimes}$ and ε for the empty multiset. We use $\sqsubseteq$ for multiset (pointwise) inclusion and $\mathscr{P}: V^* \to V^{\otimes}$ is the *Parikh* mapping that assigns a word over a finite alphabet the multiset that agrees on all multiplicities. Write *Ord* for the class of ordinal numbers.

Definition 2.1 (Basic Parallel Processes) *A* process description *is given by a finite set* $V = \{X_1,\dots,X_n\}$ *of variables, a finite set Act of actions and a finite set* T *of transition rules of the form* $X \xrightarrow{a} \alpha$ *where* $X \in V$, $a \in Act$ *and* $\alpha \in V^{\otimes}$.

A process *is a multiset in* $V^{\otimes}$ *and may be understood as the parallel composition* $X_1^{l_1}\dots X_n^{l_n}$ *of* l_1 *copies of* $X_1,\dots,$ *and* l_n *copies of* X_n. *The behavior of a process is determined by the following extension rule:*

$$\text{if } X \xrightarrow{a} \alpha \in T \text{ then } X\beta \xrightarrow{a} \alpha\beta \text{ for any } \beta \in V^{\otimes}.$$

We assume a dedicated symbol $\tau \in Act$ that is used to model *silent* steps $\xrightarrow{\tau}$ and define *weak steps* by $\overset{\tau}{\Longrightarrow} = \xrightarrow{\tau}^*$ and $\overset{a}{\Longrightarrow} = \xrightarrow{\tau}^* \xrightarrow{a} \xrightarrow{\tau}^*$ for $a \in Act \setminus \{\tau\}$. Weak steps are extended to sequences of actions inductively: for the empty word let $\Longrightarrow = \overset{\tau}{\Longrightarrow} = \xrightarrow{\tau}^*$, for non-empty sequences define $\overset{aw}{\Longrightarrow} = \overset{a}{\Longrightarrow}\overset{w}{\Longrightarrow}$ for $a \in Act, w \in Act^*$. A *deadlock* is a process that cannot make any non-silent steps. The *norm* $|\alpha|$ of a process α is length of the shortest word $w \in Act^*$ such that $\alpha \overset{w}{\Longrightarrow} \delta$ for a deadlock δ and ∞ if no such sequence exists. We call a system *normed* if all its variables have finite norm.

Definition 2.2 (Weak Bisimilarity) *A symmetric binary relation B over processes is a* weak bisimulation *iff every pair* $\alpha B \beta$ *and* $a \in Act^*$ *satisfies: if* $\alpha \xrightarrow{a} \alpha'$ *then* $\beta \overset{a}{\Longrightarrow} \beta'$ *such that* $\alpha' B \beta'$. *Two processes* α *and* β *are weakly bisimilar, denoted* $\alpha \approx \beta$, *if there exists a weak bisimulation B such that* $\alpha B \beta$.

Following [15] we characterize weak bisimilarity inductively by refinement:

Definition 2.3 (Approximants) *For a given monotone refinement function* $\Psi : 2^{V^{\otimes}\times V^{\otimes}} \to 2^{V^{\otimes}\times V^{\otimes}}$ *we define a decreasing sequence of* approximants, *subsets of* $V^{\otimes} \times V^{\otimes}$ *by transfinite induction:*

- $\approx_0 = V^{\otimes} \times V^{\otimes}$
- $\approx_{i+1} = \Psi(\approx_i)$ *for successor ordinals* $i+1$ *and*

[1] To our knowledge this conjecture appears in print only in Stríbrná's PhD thesis [19]

- $\approx_\lambda = \bigcap_{i<\lambda} \approx_i$ *for limit ordinals* λ

Weak Bisimulation approximants *are those based on the refinement function* $\mathscr{F}$ *that maps any* $R \subseteq V^{\otimes} \times V^{\otimes}$ *to the largest symmetric relation that satisfies for all* $a \in Act$ *and* $\alpha' \in V^{\otimes}$*:*

$$(\alpha,\beta) \in \mathscr{F}(R) \iff \alpha \xrightarrow{a} \alpha' \text{ implies } \exists \beta'.\beta \xRightarrow{a} \beta' \wedge (\alpha',\beta') \in R.$$

Every post-fixpoint[2] of $\mathscr{F}$ is a weak bisimulation and by a straightforward application of a fixpoint theorem due to Knaster and Tarski we see that the sequence of approximants defined by $\mathscr{F}$ converges to weak bisimilarity: $\approx = \bigcap_{o \in Ord} \approx_o$. Thus, if we have a pair of inequivalent processes α, β, then there is a least ordinal c such that $\alpha \not\approx_c \beta$. See [15], sec 4.6 for a more detailed account. Let the *convergence index* for a class of processes be the least ordinal c such that $\approx = \approx_c$ for any system of that class.

Weak bisimilarity can be characterized in terms of interactive games between two players, sometimes called Spoiler and Duplicator [17]. For a given pair of processes α and β, the game consists of a series of rounds. In each round Spoiler chooses left or right process and performs a step from it, next Duplicator must match this with an equally labeled weak step in the other process. If one of the players is not able to perform his next move then his opponent wins, infinite plays are won by Duplicator.

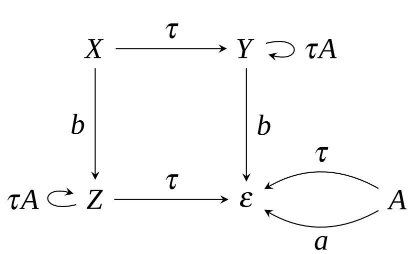
Proposition 2.4 *Two processes are weakly bisimilar iff Duplicator has a strategy to win the bisimulation game regardless of his opponents choices.*

In the same spirit we can define *approximants games* to characterize weak bisimulation approximants. A configuration of the game consist of a number $o \in Ord$ and a pair of processes α and β. In each round Spoiler chooses a new number $o' \in Ord$ such that $0 \leq o' < o$ and performs a step to α' from one of the processes. Then Duplicator responds by an equally labeled weak step from the other process to some process β'. The game continues to the next round which starts from configuration o', α', β'. If one of the players is not able to perform his next move then his opponent wins. This game cannot continue indefinitely because *Ord* is well-founded.

Proposition 2.5 *For any* $o \in Ord$ $\alpha \approx_o \beta$ *iff Duplicator has a strategy to win the approximant game from* (o,α,β) *regardless of his opponents choices.*

The intuition is that whenever Spoiler makes his move to o', α' he asserts that he can win the bisimulation game in fewer than o' rounds from the next round onwards, for any possible response of his opponent. Duplicator wins the approximants game at some limit ordinal level only if for all smaller ordinals o' he has some response that allows him to win at level o'. If in the following we write Spoiler can *distinguish* processes α and β in o rounds we mean that Spoiler wins the approximant game from (o,α,β).

Example 2.6 *Consider the process description given below, where the left-hand side is a graphical depiction of the rules listed to the right. The left shows a loop* $Y \xrightarrow{aA} Y$ *whenever there is a rule* $Y \xrightarrow{a} YA$ *in the process definition on the right-hand side.*



$$X \xrightarrow{\tau} Y, X \xrightarrow{b} Z,$$
$$Y \xrightarrow{b} \varepsilon, Y \xrightarrow{\tau} YA,$$
$$Z \xrightarrow{\tau} \varepsilon, Z \xrightarrow{\tau} ZA,$$
$$A \xrightarrow{\tau} \varepsilon, A \xrightarrow{a} \varepsilon$$

[2]an element $R \subseteq V^{\otimes} \times V^{\otimes}$ that satisfies $R \subseteq \mathscr{F}(R)$.

The two processes X and Y are inequivalent, Spoiler wins the bisimulation game by playing $(X \xrightarrow{b} Z)$; any proper response is to A^n for some n. Now Spoiler continues to play $(Z \xrightarrow{\tau} AZ \xrightarrow{a} Z)$ n times and wins in the next round. Still, Duplicator wins the approximant game from (ω, X, Y) because $ZA^i \approx_j A^j$ for any two naturals i, j and any Spoiler attack to some j, ZA^i in the fist round can be replied to by a weak step $Y \overset{\tau^j}{\Longrightarrow} YA^j \xrightarrow{b} A^j$. Hence $\approx \neq \approx_\omega$.

Example 1 shows that for the usual notion of approximants, the convergence index is above ω, so the approximation method fails. We will continue to investigate different refinement functions that yield faster converging weak bisimulation approximants.

3 Approximants

Proposition 2.5 motivates the definition of alternative refinement functions and thus approximants by changing the rules of the approximants game. That is, we define sequences of faster converging approximants by describing the abilities of the two players to move in one round of the game.

Definition 3.1 *We define different approximants by describing the way both players are allowed to move during the approximants game. In all cases Spoiler chooses the next lower ordinal and moves to some configuration, then Duplicator moves from the other process.*

Define ordinary short-long *approximants $\approx_i$ by the game in which Spoiler moves along a strong step $\xrightarrow{a}$, then Duplicator responds using a weak step $\overset{a}{\Longrightarrow}$.*

For long-long *approximants $\approx_i^L$, Spoiler makes a weak step $\overset{a}{\Longrightarrow}$, then Duplicator responds with a weak step $\overset{a}{\Longrightarrow}$.*

For word *approximants $\approx_i^W$, Spoiler moves according to a* sequence *$\overset{w}{\Longrightarrow}$ of weak steps where $w \in Act^*$, then Duplicator responds by a move $\overset{w}{\Longrightarrow}$ over the same word.*

Parikh *approximants $\approx_i^P$ are due the game where Spoiler makes a sequence of weak steps $\overset{w}{\Longrightarrow}, w \in Act^*$, then Duplicator responds by a sequence $\overset{w'}{\Longrightarrow}$ in which the letters of w are arbitrarily shuffled: $\mathscr{P}(w) = \mathscr{P}(w')$.*

Note that the short-long approximants defined here are exactly the ones given in Definition 2.3 and all others should converge faster as they give more power to Spoiler. We continue to show that all four types of approximants are indeed correct notions of approximation for weak bisimilarity and do not converge towards something even smaller in the limit. Afterwards, we look at how suitable they are for the approximation method we have in mind.

Lemma 3.2 *For any ordinal i, $\approx \subseteq \approx_i^W \subseteq \approx_i^P \subseteq \approx_i^L \subseteq \approx_i$.*

proof For the first inclusion assume that (α, β) is in $\approx$, so there is a weak bisimulation B containing this pair. This means for any move $\alpha_0 \xrightarrow{a_1} \alpha_1 \xrightarrow{a_2} \ldots \xrightarrow{a_k} \alpha_k, a_j \in Act$ there is a sequence $\beta_0 \overset{a_1}{\Longrightarrow} \beta_1 \overset{a_2}{\Longrightarrow} \ldots \overset{a_k}{\Longrightarrow} \beta_k$ with $\alpha_j B \beta_j$ for $j \leq k$, so B prescribes a winning strategy for Duplicator in the word-approximant game.

For the second inclusion observe that if Duplicator has a response $\beta \overset{w}{\Longrightarrow} \beta'$ for some attack $\alpha \overset{w}{\Longrightarrow} \alpha'$ clearly the same response is allowed in the game where he may arbitrarily shuffle the letters of w.

For the third inclusion assume $(\alpha, \beta) \notin \approx_i^L$, then Spoiler can distinguish the two processes in i rounds where he only uses weak steps $\overset{a}{\Longrightarrow}$ labelled by single actions and his opponent may also respond using equally labelled weak steps. But the same strategy will be winning for Spoiler if he is allowed to make steps $\overset{w}{\Longrightarrow}$ due to sequences of actions and his opponent may arbitrarily shuffle the actions in his response: If an attack is due to a single action the response must be due to a single action. Thus Spoiler can distinguish (α,β) in at most i rounds of this Parikh-game: $(\alpha,\beta) \notin \approx_i^P$. The last inclusion follows similarly: If Spoiler can distinguish two processes in i rounds if he is only allowed to make strong steps $\xrightarrow{a}$ and his opponent can do weak steps as response, then must also be able to distinguish the processes in at most i rounds of a game in which he can also make weak attacks. ∎

Theorem 3.3 $\approx = \bigcap_{i\in Ord} \approx_i^W = \bigcap_{i\in Ord} \approx_i^P = \bigcap_{i\in Ord} \approx_i^L = \bigcap_{i\in Ord} \approx_i$

proof The chain of inclusions $\subseteq$ holds by transfinite induction using Lemma 3.2. Milner [15] shows that sequence of short-long approximants converges to weak bisimilarity: $\approx = \bigcap_{i\in Ord} \approx_i$. ∎

Lemma 3.4 *For any BPP description and ordinal i we have*

1. $\approx_i^L, \approx_i^P, \approx_i^W$ *are equivalences and*
2. *for* $\sim \in \{\approx_i, \approx_i^L, \approx_i^P, \approx_i^W\}$ *it holds that* $\alpha \sim \beta$ *implies* $\alpha\gamma \sim \beta\gamma$.

proof 1. Let $O \in \{L,P,W\}$. We show transitivity by induction: $\approx_0^O = V^\otimes \times V^\otimes$ is trivially transitive. Assume $\approx_i^O$ is transitive for $i \in Ord$ and 1) $\alpha \approx_{i+1}^O \beta$ and 2) $\beta \approx_{i+1}^O \gamma$. We show that Duplicator wins the O-approximants game that starts at $(i+1,\alpha,\gamma)$. Without loss of generality one can assume that Spoiler moves $\alpha \stackrel{u}{\Longrightarrow} \alpha'$. By 1) we know that in the game α vs. β there is a valid response $\beta \stackrel{v}{\Longrightarrow} \beta'$ such that $\alpha' \approx_i^O \beta'$. Equally well if in the game β vs. γ, Spoiler moves $\beta \stackrel{v}{\Longrightarrow} \beta'$ then by 2) there is a valid response $\gamma \stackrel{w}{\Longrightarrow} \gamma'$ with $\beta' \approx_i^O \gamma'$. By induction hypotheses we have $\alpha \approx_i^O \gamma$, so by definition of $\approx_{i+1}^O$ also $\alpha \approx_{i+1}^O \gamma$.

For limit ordinals l this goes analogously: for Spoilers attack from α there is a response from β for all smaller ordinals i; for any such move there is a response from γ to some process equivalent at level i. By assumption $\alpha \approx_i^O \gamma$ and hence $\alpha \approx_l^O \gamma$ by definition. Symmetry and reflexivity follow trivially from the definition.

The second claim is a result of Duplicator using a strategy that remembers which parts of the configurations $\alpha\gamma, \beta\gamma$ come from α,β and γ. Every move of Spoiler from $\alpha\gamma$ (or $\beta\gamma$) can be split into two parts, one which originates from α (or β) and the one which was performed from variables that come from γ. Duplicators response will be the combined responses for the first and the second part of Spoilers attack in the games α vs. β and γ vs. γ. In the second part Duplicator simply copies Spoilers move and can therefore even preserve equality on the parts of the processes that derive from γ. This means Spoiler cannot distinguish $\alpha\gamma$ and $\beta\gamma$ in fewer rounds than he can distinguish α and β. ∎

The first claim of the lemma does not hold for the short-long approximants $\approx_i$ because Spoiler and Duplicator have different abilities to move. For a counter-example to their transitivity consider example below.

Example 3.5 *The following rules describe a system with* $X \approx_1 Y \approx_1 Z \not\approx_1 X$:

$$Y \xrightarrow{\tau} X,\; Y \xrightarrow{\tau} Z,\; Y \xrightarrow{\tau} Y',\; Y' \xrightarrow{a} \varepsilon,\; Y' \xrightarrow{b} \varepsilon,\; X \xrightarrow{a} X,\; Z \xrightarrow{b} Z.$$

We will continue to show that for finite ordinals $i < \omega$, the approximants $\approx_i, \approx_i^L$ and $\approx_i^P$ are decidable. For this we recall *Presburger Arithmetic*, the first order logic of natural numbers with addition and equality. Syntactically, a Presburger Arithmetic formula is $True, False$, a statement $t_1 = t_2$ where the terms t_1,t_2 are sums of natural numbers or variables, any boolean combination of smaller formulae or a universally or existentially quantified formula. We write $F(x_1,x_2\dots x_k)$ for the formula F in which the variables $x_1 \dots x_k$ occur freely, i.e. not in the scope of a quantifier and interpret formulae over natural numbers and equality. A set $R \subseteq \mathbb{N}^k$ of k-tuples of natural numbers is said to be *Presburger-definable* if there is a Presburger Arithmetic formula $\Phi_R(x_1 x_2 \dots x_k)$ that satisfies

$$\Phi_R(x_1 x_2 \dots x_k) \equiv True \iff (x_1 x_2 \dots x_k) \in R$$

An important property of Presburger Arithmetic is that it is decidable if a given a Presburger Arithmetic formula Φ without free variables is True. This implies that Presburger-definable sets are decidable. Moreover, the class of Presburger-definable sets coincides with the class of *semi-linear* sets [6] which for our purposes means it is effectively closed under projection and intersection. We refer to [6] for the details on Presburger Arithmetic.

Any relation R over BP processes with k variables is a subset of $\mathbb{N}^{2k}$. We now show that for finite n, the approximants $\approx_n, \approx_n^L$ and $\approx_n^P$ are effectively Presburger-definable and therefore decidable relations. We recall an important result from [5], Thm 3.3:

Lemma 3.6 *For any BPP description, the set Reach* $\subseteq V^{\otimes} \times Act^{\otimes} \times V^{\otimes}$ *of triples* (α, μ, β) *such that* $\alpha \xrightarrow{a_1} \alpha_1 \xrightarrow{a_2} \alpha_2 \dots \xrightarrow{a_n} \beta$ *for some sequence* $a_1 a_2 \dots a_n \in Act^*$ *with* $\mathscr{P}(a_1 a_2 \dots a_n) = \mu$ *is effectively Presburger-definable.*

From this we can conclude that the step and weak step relations $\xrightarrow{a}, \xRightarrow{a}$ are effectively Presburger-definable: The sets $S_1 = \{a\}$, and $S_2 = \{a\}\{\tau\}^{\otimes}$ (in other words the Parikh images of $a\tau^*$) are easily seen to be Presburger-definable and $\xrightarrow{a}$ and $\xRightarrow{a}$ are expressible as the projections into the first and third component of $Reach \cap (V^{\otimes} \times S_1 \times V^{\otimes})$ and $Reach \cap (V^{\otimes} \times S_2 \times V^{\otimes})$ respectively.

Theorem 3.7 *For a given BP process description B with k variables the n-th approximants* $\approx_n, \approx_n^L$ *and* $\approx_n^P$ *over B are decidable for all finite n.*

proof It suffices to to show that $\approx_n, \approx_n^L$ and $\approx_n^P$ are effectively Presburger-definable. By Lemma 3.6 we can assume a Presburger Arithmetic formula $R \subseteq \mathbb{N}^V \times \mathbb{N}^{Act} \times \mathbb{N}^V$ that expresses the set *Reach* and formulae $Step_a, WStep_a \subseteq \mathbb{N}^V \times \mathbb{N}^{Act} \times \mathbb{N}^V$ expressing the strong and weak a-step relations for all actions $a \in Act$. Now we can easily encode the refinement functions used in the approximants and for any finite n construct the Presburger Arithmetic formulae that express $\approx_n, \approx_n^L$ and $\approx_n^P$ by induction:

For $n = 0$ we have $\approx_0 = \approx_0^L = \approx_0^P = \mathbb{N}^{2k}$ trivially definable as $\Psi_0(\alpha, \beta) = True$.

For $\approx_{i+1}$ let $\Psi_{i+1}(\alpha, \beta) \iff \bigwedge_{a \in Act}$ (

$$\begin{aligned} &(\forall \alpha' \in \mathbb{N}^V\ Step_a(\alpha, \alpha') \implies \exists \beta' \in \mathbb{N}^V\ WStep_a(\beta, \beta') \wedge \Psi_i(\alpha', \beta')) \\ \wedge &(\forall \beta' \in \mathbb{N}^V\ Step_a(\beta, \beta') \implies \exists \alpha' \in \mathbb{N}^V\ WStep_a(\alpha, \alpha') \wedge \Psi_i(\alpha', \beta'))) \end{aligned}$$

Similarly, for $\approx_{i+1}^L$ let $\Psi_{i+1}(\alpha, \beta)$ as above but replace $Step_a$ by $WStep_a$. For $\approx_{i+1}^P$ let $\Psi_{i+1}(\alpha, \beta) \iff \forall \mu \in \mathbb{N}^{Act}$ (

$$\begin{aligned} &(\forall \alpha' \in \mathbb{N}^V\ R(\alpha, \mu, \alpha') \implies \exists \beta' \in \mathbb{N}^V\ R(\beta, \mu, \beta') \wedge \Psi_i(\alpha', \beta')) \\ \wedge &(\forall \beta' \in \mathbb{N}^V\ R(\beta, \mu, \beta') \implies \exists \alpha' \in \mathbb{N}^V\ R(\alpha, \mu, \alpha') \wedge \Psi_i(\alpha', \beta'))) \end{aligned}$$

∎

It is worth mentioning that word approximants $\approx_n^W$ are not decidable at finite levels: for systems without silent actions the very first approximant $\approx_1^W$ coincides with *trace equivalence*, which has been shown to be undecidable for BPP by Hirshfeld [9].

4 Applications

We now use the approximation approach to show that two subclasses of BPP previously known in the literature have decidable weak bisimilarity. In particular, we show this result in Section 4.1 for the class introduced in [18] by proving weak bisimilarity finitely approximable for the long-long approximants $\approx^L$ and for the subclass introduced in [19] we show finite approximability for Parikh approximants $\approx^P$ in Section 4.2. In both cases we know by Theorem 3.7 that at finite levels the approximants are decidable equivalences and hence showing their convergence at level ω suffices to get a decision procedure.

Proposition 4.1 *The following states some useful facts that are easily verified.*

1. $\alpha \approx \beta$ *implies* $|\alpha| = |\beta|$
2. *If* $\alpha \Longrightarrow \beta \Longrightarrow \alpha'$ *and* $\alpha \approx \alpha'$ *then* $\alpha \approx \beta$.
3. *If* $\alpha \Longrightarrow \alpha\beta$ *and* β *has norm* 0 *then* $\alpha \approx \alpha\beta$

Definition 4.2 *Let* $O \in \{L,P,W\}$ *and* $\alpha,\beta \in V^{\otimes}$ *such that* $\alpha \approx^O_\omega \beta$. *For a given Spoiler move from* α *to* α' *there is a sequence* $B = \beta'_1, \beta'_2, \beta'_3 \ldots$ *of Duplicator responses such that for all* $i \in N$ *holds* $\alpha' \approx^O_i \beta'_i$. *We call B a* family of responses.

Observe that the sequence is not unique, for example if you substitute β_i by β_j for any $j > i$ then you obtain another family of responses. By Dickson's Lemma we can assume that a family of responses is non-decreasing with respect to multiset inclusion: $\beta_i \sqsubseteq \beta_{i+1}$ for every $i \in \mathbb{N}$.

4.1 Normed Processes with Pure Generators

Write $\alpha \longrightarrow_0 \beta$ for silent and norm-preserving steps between processes $\alpha,\beta \in V^{\otimes}$: $\alpha \longrightarrow_0 \beta$ iff $\alpha \xrightarrow{\tau} \beta$ and $|\alpha| = |\beta|$. Let $\Longrightarrow_0$ be the transitive and reflexive closure of $\longrightarrow_0$. For variables X,Y such that $X \Longrightarrow_0 Y \Longrightarrow_0 X$ we have $X \approx Y$ by Claim 2) Proposition 4.1. We say X is *redundant* because of Y or vice versa. One can easily detect redundant variables and therefore we can assume that they have already been unified. That is, we can assume wlog. that our process description does not contain redundant variables. This allows us to linearly order the set V of variables such that if $X \Longrightarrow_0 Y\alpha$ then $X > Y$. Let's fix the notation $X_1 > X_2 > \ldots > X_k$.

A *generator* is a variable X that allows a sequence $X \Longrightarrow_0 X\alpha$ for some $\alpha \in V^{\otimes}$, in which case we say X *generates* α. Call a generator X *pure* if $X \Longrightarrow_0 \alpha$ implies that $\alpha = \alpha'X$: Pure generators cannot vanish silently.

Stirling shows decidability of weak bisimilarity for normed processes with only pure generators using a tableaux approach [18]. One motivation for this subclass is that it still allows for infinite branching and that ordinary ($\approx_i$) approximants do not converge at level ω. In this section we show that long-long ($\approx^L_i$) approximants in fact stabilize at level ω and thus provide the missing negative semidecision prodecure to conclude decidability.

Lemma 4.3 *Let* α *be a normed process of a BPP description without redundant variables in which every generator is pure.* $Succ = \{\alpha' | \alpha \Longrightarrow_0 \alpha'\}$ *can be partitioned into finitely many equivalence classes with respect to weak bisimilarity.*

proof The third claim of Proposition 4.1 allows us to restrict ourselves to the subset $Succ'$ of $Succ$ of configurations which are obtained without use of generating moves because it has the same number equivalence classes as generators cannot vanish along $\Longrightarrow_0$ moves. Our goal is to show that $Succ'$ is finite which immediately implies the claim of the lemma.

Every derivation of α is a sum of derivations from variables belonging to α. If we prove that in silent norm preserving steps without generating moves, we can only derive finitely many configurations from each variable, then we will also prove that $Succ'$ is finite. We will show that this is indeed the case for all variables by induction over the assumed order $<$. From the smallest variable X_k using silent norm preserving steps *without generating* we can derive only two configurations, namely X_k or ε.

Assume $c > 0$ bounds the number of possible silent norm preserving derivations from any variable in $X_i \dots X_k$ and consider the variable X_{i-1}. In case X_{i-1} is a deadlock variable, i.e. $X_{i-1} \xrightarrow{\tau} X_{i-1}$ is the only applicable rule, we can trivially bound the number of its derivations by $1 \leq c$. Otherwise, because we forbid generating moves we must have that any rule $X_{i-1} \xrightarrow{\tau}_0 \alpha$ produces a multiset $\alpha \in \{X_i \dots X_k\}^{\otimes}$. The fact that there are only finitely many rules that rewrite variable X_{i-1} implies that we can bound the number of its silent norm preserving derivations by

$$d \cdot c^l + 1,$$

where d is the number of rules for X_{i-1} and l is the maximal size of any right hand side of a rule rewriting X_{i-1}. ∎

Theorem 4.4 $\approx = \approx^L_\omega$ *for normed BPP where each generator is pure.*

proof Assume towards a contradiction that we have $\alpha \approx^L_\omega \beta \not\approx^L_{\omega+1} \alpha$. Wlog. assume an optimal[3] initial move $\alpha \overset{a}{\Longrightarrow} \alpha'$ for Spoiler in the game α vs. β and a family $B = \beta'_0, \beta'_1, \dots$ of responses which is strictly increasing wrt. multiset inclusion.

By Lemma 4.3, the set $Succ = \{\alpha'' | \alpha' \Longrightarrow_0 \alpha''\}$ of configurations reachable from α' in silent and norm-preserving steps contains finitely many bisimilarity classes. Let the set $Succ'$ be a finite set of representants of those classes in $Succ$. This allows us to define a function $f : B \to Succ'$ that maps $\beta'_i \in B$ to an element in $Succ'$ that maximises their approximation index: $\beta'_i \approx^L_k f(\beta'_i)$ and $\forall \gamma \in Succ'\ \beta'_i \approx^L_l \gamma \Longrightarrow k \geq l$. This function is well defined because set $Succ'$ is finite. Now consider an infinite subsequence $B(\gamma)$ of B that contains all elements which f maps to the configuration $\gamma \in Succ'$. By the pigeon hole principle such a subsequence exists.

Take two different elements $\beta'_i \sqsubset \beta'_j$ of $B(\gamma)$ for arbitrary large i, j. We have 1) $\beta'_i \approx^L_i \gamma \approx^L_j \beta'_j$ because $\alpha' \in Succ'$ and 2) β'_i and β'_j have the same norm. To see why the second obervation is true note that $|\alpha| \neq |\beta|$ implies $\alpha \not\approx^L_{\min\{|\alpha|,|\beta|\}} \beta$ as Spoiler only needs to decrease the smaller process to a deadlock which cannot be mimiked by Duplicator on the other process because the norms differ. We know $\beta'_i \approx^L_i \alpha' \approx^L_j \beta'_j$, so $|\beta'_i| = |\alpha'| = |\beta'_j|$ as otherwise i and j would be bounded by $|\alpha'|$.

Consider the game on α' vs. β'_j and a silent, norm-preserving move $\beta'_j \Longrightarrow_0 \beta'_i$ made by Spoiler, which must be possible due to observation 2) and the fact that β'_i is a subset of β'_j. Now by definition of the subsequence $B(\gamma)$ we deduce that $\alpha' \Longrightarrow_0 \gamma$ is an optimal response for Duplicator. Therefore by 1), we know that $\beta'_i \approx^L_{j-1} \gamma$ so $\beta'_i \approx^L_{j-1} \beta'_j$ by transitivity and the fact that $\beta'_j \approx^L_{j-1} \gamma$. But now we have $\beta'_i \approx^L_{j-1} \alpha'$ for arbitrarily high j and therefore $\beta'_i \approx^L_\omega \alpha'$ which contradicts the optimality of Spoiler's very first move. ∎

4.2 Unnormed Processes over one visible Action

Consider the subclass of BPP processes that satisfy both

[3]a move prescribed by an optimal winning strategy: one that guarantees a win for Spoiler in the fewest number of rounds and thus properly decreases the approximation index in each round.

1. There is only one visible action label, $Act = \{\tau, a\}$ and
2. Every variable has positive or infinite norm.

This class has been introduced in [19], where it was shown that for processes of this kind, Hirshfelds conjecture holds: $\approx = \approx^L_{\omega*2}$. Note that this class is not a subclass of the *totally normed* systems [10] as it allows for variables with infinite norm. We show that this class has decidable weak bisimilarity by showing that Parikh-approximants converge at level ω.

Theorem 4.5 $\approx = \approx^P_\omega$ *for the subclass of BPP processes with a single visible action and no variables with norm* 0*.*

proof First observe that 1) implies that all configurations with infinite norm must be equivalent and due to norm preservation cannot be equivalent to any configuration of finite norm. The second restriction guarantees that there are only finitely many different configurations for any given finite norm. 2) Whenever two processes have different but finite norms, they are certainly not related by $\approx^P_2$ as Spoiler may rewrite the smaller process to a deadlock in one long step without allowing his opponent to do the same on the other process.

Assume towards a contradiction that $\alpha \approx^P_\omega \beta \not\approx^P_{\omega+1} \alpha$. So for an optimal initial move $\alpha \stackrel{w}{\Longrightarrow} \alpha'$ for Spoiler there is a family of responses from β. This sequence cannot converge as otherwise our assumption $\beta \not\approx^P_{\omega+1} \alpha$ would be false. By the pidgin hole principle, there must be at least one variable X that grows indefinitely along this sequence. Take two elements $\beta'_i \sqsubset \beta'_j$, $2 < i < j$ from this sequence such that X occurs more often in β'_j. By observation 2) and the fact that β'_i and β'_j have different norms we know that $\beta'_i \not\approx^P_2 \beta'_j$. Because $\beta'_i \approx^P_i \alpha' \approx^P_j \beta'_j$ and $i < j$ holds $\beta'_i \approx^P_i \alpha' \approx^P_i \beta'_j$. From this and the transitivity of $\approx^P_i$ we conclude that $\beta'_i \approx^P_i \beta'_j$ and because $2 < i$ also $\beta'_i \approx^P_2 \beta'_j$ which is a contradiction. ■

5 Limitations of the Approximant Approach

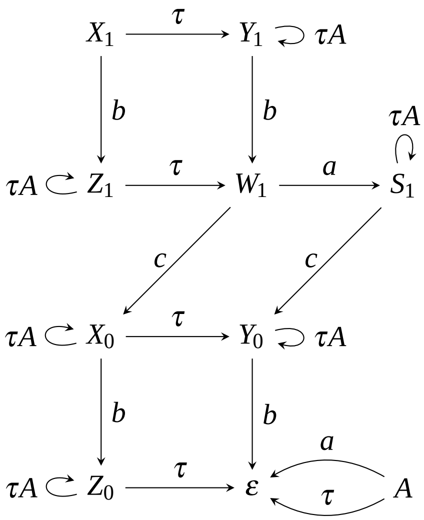
One severe limitation of the approximation method is that it cannot provide complexity bounds even if successfully applied. In this section we show that $\approx^L$ is not guaranteed to stabilize at level $\omega * 2$ and that word approximants $\approx^W$ do not necessarily stabilize on level ω. From our counter-examples we derive lower bounds of ω^2 and $\omega * 2$ for the convergence indices of $\approx^L$ and $\approx^W$ respectively.

Theorem 5.1 *Long-Long approximants* ($\approx^L_i$) *do not stabilize below level* ω^2 *for BPP:* $\approx \neq \approx^L_{\omega*k}$ *for all finite k.*

proof For $k < 2$ the claim is trivial, e.g. by Example 2.6. We first show how to construct a system with $\approx \neq \approx^L_{\omega+\omega}$. For this we recycle Example 2.6 and add the rule $X \stackrel{\tau}{\longrightarrow} XA$ and analyze the game on X vs. Y more carefully. The fact that X can be silently rewritten to Y forces Spoiler to start from X. Any optimal silent move for Spoiler must change the equivalence class, so we can assume his initial move to be $X \stackrel{b}{\Longrightarrow} ZA^m$. Duplicator must respond to some A^n. To prevent a perfect match to an identical process in the next round, Spoiler must again move from ZA^m and may not end in a configuration $A^{<n}$. So Spoiler will either move $Z \stackrel{a}{\Longrightarrow} Z$ or $Z \stackrel{a}{\Longrightarrow} A^m$ with $m \geq n$ and thereby force Duplicator to remove one A on the other side. Observe that any one move from Z or A^m can be replied to by A, so Spoiler has to keep making a-moves from his process until Duplicator has exhausted all variables A. By removing only one A in each such response, Duplicator can prevent the situation Z (or $A^{>0}$) vs. ε for n rounds, where n is determined by his initial response. We conclude $X \approx^L_\omega Y \not\approx X$.

To construct a counter-example to convergence at level $\omega + \omega$ we combine two copies of this system as indicated in Figure 1 below.

Figure 1: Combining two copies of the "Guessing Game" yields $X_1 \approx^L_{\omega*2} Y_1 \not\approx X_1$.



$$X_1 \xrightarrow{\tau} Y_1 \quad X_1 \xrightarrow{b} Z_1,$$
$$Y_1 \xrightarrow{b} W_1, \quad Y_1 \xrightarrow{\tau} Y_1A,$$
$$Z_1 \xrightarrow{\tau} W_1, \quad Z_1 \xrightarrow{\tau} Z_1A,$$
$$W_1 \xrightarrow{a} S_1, \quad S_1 \xrightarrow{\tau} S_1A,$$
$$W_1 \xrightarrow{c} X_0, \quad S_1 \xrightarrow{c} Y_0$$

$$X_0 \xrightarrow{\tau} Y_0, \quad X_0 \xrightarrow{b} Z_0,$$
$$Y_0 \xrightarrow{b} W_0, \quad X_0 \xrightarrow{\tau} X_0A,$$
$$Y_0 \xrightarrow{\tau} Y_0A, \quad Z_0 \xrightarrow{\tau} Z_0A,$$
$$Z_0 \xrightarrow{\tau} \varepsilon$$

$$A \xrightarrow{\tau} \varepsilon \quad A \xrightarrow{a} \varepsilon,$$

The bottom part of the construction is the gadget as discussed previously. Observe that variables X_0, Y_0, Z_0 are not able to produce variables from the top part of the diagram, those variables with an index 1. Thus we preserve that $X_0 \approx^L_\omega Y_0$. Our aim is to show that indeed $X_1 \approx^L_{\omega+\omega} Y_1 \not\approx X_1$. For this it suffices to show that the only possibility for Spoiler to win is to force the game from X_1 vs. Y_1 to end up in $X_0 \approx^L_\omega Y_0$.

The Game starts from a pair X_1, Y_1 and it goes through the upper square pattern X_1, Y_1, Z_1, W_1. By our previous discussion of this gadget, we know that Spoiler has to start by $X_1 \stackrel{b}{\Longrightarrow} Z_1A^m$; Duplicator will respond to W_1A^n. Spoiler must continue to play from the left hand side in order to prevent a perfect match to identical processes and cannot move to a W_1A^i for $i \leq n$. If he makes a move $Z_1A^m \xrightarrow{c} X_0A^i$, while the other process still contains a W_1, Duplicator is able to match to the same process. So the only option left for Spoiler is to force Duplicator to remove all variables A one by one by performing a-steps. Eventually, from a position Z_1 (or $W_1A^{>0}$) vs. W_1, Spoiler makes one last a-step and thus forces Duplicator to rewrite W_1 to S_1. Afterwards, Spoiler can force the game to a position X_0A^n vs. Y_0A^m by playing a c-step from either side. This part of the game takes $n+1$ rounds and n was chosen by Duplicator in his first response. Therefore $X_1 \approx_{\omega+\omega} Y_1$ which completes the proof for $k = 2$.

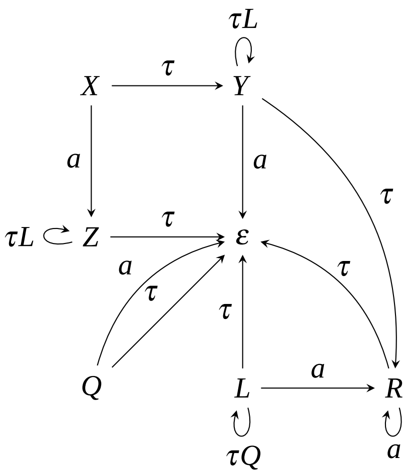
The construction above can be extended to provide a counter-example for convergence at level $\omega * k$ for any natural k by stacking k copies of the square gadget on top of each other. This can also be modified to a system which contains only variables of the norm zero. ∎

Next we focus on Word approximants and falsify a conjecture of Stríbrná [19] about their convergence above level ω.

Theorem 5.2 *For BPP, weak bisimilarity is not finitely approximable with word approximants:* $\approx \neq \approx^W_\omega$.

proof Consider the process description in Figure 2. By Proposition 4.1 part 3, we know that $ZL^nQ^m \approx Z$ and $L^{n+1}Q^m \approx L^{n+1}$ for any two naturals m, n. We claim that $X \approx^W_\omega Y \not\approx X$ and base our proof on the

Figure 2: Counter-example for finite approximability of $\approx_i^W$



$$
\begin{array}{ll}
X \xrightarrow{a} Z, & X \xrightarrow{\tau} Y \\
Y \xrightarrow{a} \varepsilon, & Y \xrightarrow{\tau} YL \\
Y \xrightarrow{\tau} R & \\
Z \xrightarrow{\tau} ZL\,, & Z \xrightarrow{\tau} \varepsilon \\
L \xrightarrow{a} R, & L \xrightarrow{\tau} \varepsilon \\
L \xrightarrow{\tau} LQ & \\
R \xrightarrow{\tau} \varepsilon\,, & R \xrightarrow{a} R \\
Q \xrightarrow{a} \varepsilon, & Q \xrightarrow{\tau} \varepsilon
\end{array}
$$

following claims that are proven individually after the main argument. For $i, j, n \in \mathbb{N}$, $n > 0$ we have

$$Z \not\approx_3^W RL^i \not\approx_3^W L^j, \tag{1}$$

$$Z \approx_{2n+1}^W L^n, \tag{2}$$

$$Z \not\approx_{2n+2}^W L^n. \tag{3}$$

In the game X vs. Y Spoiler must start with a move $X \stackrel{a}{\Longrightarrow} ZL^lQ^q \approx Z$, as otherwise his opponent is able to match to the same process and thereby win. Possible responses for Duplicator from Y are:

- To some $RL^nQ^m \approx RL^n$, which allows Spoiler to win in 3 further rounds by Claim 1.
- To some $YL^nQ^m \approx YL^n$ which allows Spoiler to silently replace the Y by R and afterwards again win in 3 rounds by Claim 1). Note that no silent response from ZL^lQ^q to some configuration that contains R is possible.
- To some Q^m which allows Spoiler to win in one round by playing $Z \stackrel{a^{m+1}}{\Longrightarrow} Z$.
- To some $L^nQ^m \approx L^n, n > 0$ which allows Spoiler to win but in not fewer than $2n+2$ rounds by Claims 2) and 3).

The choice of n is made by Duplicator and therefore $X \approx_\omega^W Y \not\approx X$. Note that this counter-example uses only a single visible action and all variables have zero norm. ∎

It remains to proof claims 1.-3. We first prove some auxiliary claims on which we base our arguments for claims 1) and 2). For all $m, n \in \mathbb{N}$,

$$RL^n \not\approx_1^W Q^m \tag{4}$$

$$L^n \not\approx_2^W R \not\approx_2^W Z \tag{5}$$

For (4), observe that Duplicator cannot respond to a move $R \stackrel{a^{m+1}}{\Longrightarrow} R$. For (5), Spoiler moves from L^n (or Z) silently to Q and Duplicator can respond to R or to ε. In the first case he loses in one round by claim (4), in the latter he cannot respond to move $Q \stackrel{a}{\Longrightarrow} \varepsilon$ from ε.

Claim (1): $Z \not\approx^W_3 RL^n \not\approx^W_3 L^m$**.**

proof For both parts Spoiler moves from RL^n silently to R. Duplicator must respond either to Q^k which is losing for him in one round by Claim (4), or to $L^jQ^k \approx L^j$ or $ZL^jQ^k \approx Z$, which is losing for him in two rounds by Claim (5). ∎

Claim (2): $Z \approx^W_{2n+1} L^n$ **for** $n > 0$**.**

proof By induction on $n \geq 1$ together with the claim that for any $m > n$, $L^m \approx^W_{2n+1} L^n$. Base case $L \approx^W_3 L^m$. Wlog. assume that m minimizes k in $L \approx^W_k L^m$ and that Spoiler only makes optimal moves i.e. wins as quickly as possible. This means in particular that he needs to change the equivalence class in every move. Thus, he can move either $L^m \stackrel{a^*}{\Longrightarrow} L^{m'}Q^q \approx L^{m'}$ or to $L^{m'}RQ^q$ for some $l < m$. In both cases Duplicator to stays in L. In the first case, because we assume optimal moves, we must have $L \not\approx^W_i L^l$ for some $i < k$, which contradicts the optimality of m. Alternatively, the game continues from L^lRQ^q vs. L. Spoiler must again move from L^lRQ^q and change the class. If he makes an a-step to R or ends in Q^i Duplicator can match to the same process, a move to some $RL^{l'}$ or $L^{l'}, l' < l \leq m$ is surely non-optimal. The only remaining option is to move silently to R to which Duplicator will respond by $L \Longrightarrow L$. Now observe that $L \approx^W_1 R$.

Base case $Z \approx^W_3 L$: As Z can silently go to L^n, Spoiler needs to start from Z. He has three options to change the class from here: to some $L^lQ^q \approx L^l$, to $RL^lQ^q \approx RL^l$ or to something equivalent to ZR. In all cases Duplicator responds to L and in the first two cases, we can use provious claims $L \approx^W_3 L^m$ and $L \approx^W_2 RL^m$ to conclude that this allows him to survive another 2 rounds. If the second round starts in L vs. ZR (or equivalent), Spoiler can again not move from L and has three options to change the class: to something equivalent to Z which is non-optimal as it repeats the initial configuration. Alternatively he can go to $RL^lQ^1 \approx RL^l$ or to $RQ^q \approx R$. In both cases we complete by the observation that $RL^l \approx^W_1 L \approx^W_1 R$.

For the induction step, assume $L^m \approx^W_{2n+1} L^n$ and $Z \approx^W_{2n+1} L^n$. We show that $L^m \approx^W_{2(n+1)+1} L^{(n+1)}$: Just as in the base case, the only good move for Spoiler is $L^m \stackrel{a}{\Longrightarrow} L^{m'}RQ^q$ for some $n < m' < m$. Duplicator in his response goes to L^nR. Next one more time Spoiler has the only one reasonable kind of move, to a process equivalent to $L^{m''}$, where $m'' > n$. However now Duplicator responds to L^n and we use the induction assumption to the pair $L^{m''} \approx^W_{2n+1} L^n$.

Observe that because $\approx^W_{2n+1}$ is a congruence this implies also $L^mR \approx^W_{2n+1} L^nR$ for $m \geq n$. To show that $Z \approx^W_{2(n+1)+1} L^{(n+1)}$ we assume wlog. that Spoiler initially moves $Z \stackrel{a}{\Longrightarrow} ZR$, Duplicator responds by $L^{n+1} \stackrel{a}{\Longrightarrow} L^nR$. Now to prevent a perfect match in the next round, Spoiler moves from ZR to either Z or to L^mR or L^m. In the first case, Duplicator will remove the R and end up in L^n and we can use the induction assumption, in the last two cases Duplicator stays in L^nR or goes to L^n. Either way, we can use the previous claims that $L^mR \approx^W_{2n+1} L^nR$ and $L^m \approx^W_{2n+1} L^n$ for $m \geq n$. ∎

Claim (3): $Z \not\approx^W_{2n+2} L^n$ **for** $n > 0$**.**

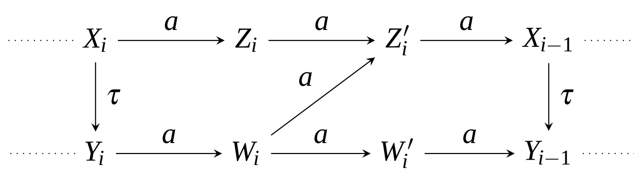
proof By induction on $n \geq 1$ together with the claim that for any $m > n$, $L^m \not\approx^W_{2n+2} L^n$. Base case: $Z \not\approx^W_4 L \not\approx^W_4 L^m$. Spoiler plays $L^m \stackrel{a}{\Longrightarrow} LR$ (or $Z \stackrel{a}{\Longrightarrow} LR$). Possible responses from L are

1. to LQ^q or Q^q, from which Spoiler wins in 3 rounds by Claim 1.
2. to RQ^q in this case Spoiler performs a move $LR \stackrel{\tau}{\Longrightarrow} Q^{q+1}$ and Duplicator responds to either RQ^i or Q^i with $i \leq q$. In both cases Spoiler wins in one round by claim (4) or playing an a^{q+1}-step from Q^{q+1}.

For the induction step we assume $L^m \not\approx^W_{2n+2} L^n \not\approx^W_{2n+2} Z$ and show that both $L^m \not\approx^W_{2(n+1)+2} L^{(n+1)}$ and $Z \not\approx^W_{2(n+1)+2} L^{(n+1)}$ hold. Spoiler moves from L^m (or Z) in an a-step to $L^{n+1}R$. Duplicator can respond

1. to $L^n Q^q$ or some Q^q, from which Spoiler wins in 3 rounds by Claim 1.
2. to $L^{n'} R Q^q, n' < n$. In this case Spoiler performs a move $L^n R \stackrel{\tau}{\Longrightarrow} L^n$ and Duplicator responds either to $L^{n''} R Q^i$ or $L^{n''} Q^i$ with $n'' \leq n' < n$. In the first case, Spoiler wins in one round by claim (4). In the last case the game continues from L^n vs. $L^{n''<n}$ and we can use the induction assumption. ∎

Remark 5.3 *To construct a counter-example to convergence of word approximants at level $\omega + k$ for finite k, the previous construction can be complemented by a "finite ladder", where X and Y are renamed to X_0 and Y_0: For $0 < i \leq k$ add variables $X_i, Y_i, Z_i, Z'_i, W_i, W'_i$ and rules as indicated below.*



6 Discussion

In order to decide weak bisimulation for BPP or subclasses it suffices to provide a semi-decision procedure for inequivalence. If we have some measure on which equivalent processes must agree, we can define a new notion of approximants by additionally requiring that Duplicator must preserve equality on this measure in every round of an approximation game. Conversely, one can think of properties as being captured by some notion of approximation $\approx^O$: If two processes disagree on the property then they are distinguished by $\approx^O_i$ at some level $i \leq \omega$.

As an example take the property *norm preservation* of Claim 1) Proposition 4.1: Equivalent processes must have equal norms. This is captured by Parikh or Word approximants because if two processes disagree on the norm, Spoiler can distinguish them in two rounds of the corresponding game by reducing the smaller one to a deadlock – which cannot be done in any proper response from the other side – and playing an action from the non-deadlocked process afterwards. Another known invariant are the *distance to disabling* functions (dd-functions) used in [14] for strong bisimulation. If the shortest path from α to α' which disables any action a is shorter than a shortest path from β to a configuration which disables a then $\alpha \not\approx^P_2 \beta$. So this first level of dd-functions is captured by Parikh approximants at level 2. We can continue this argument and say n-th order dd-functions are capture by $\approx^P_{n+1}$ relation.

We have shown that all subclasses of BPP which are currently known to have decidable weak bisimulation are indeed finitely approximable for some natural notion of approximation. The lower bound of $\omega + \omega$ for the convergence of Word (and thus Parikh) approximants given by the construction in Theorem 5.2 leads us to the conclusion that we are in fact looking for a distinguishing property that is orthogonal to Word approximants: It should still allow for decidable approximants but at the same time it must be stronger than (not captured by) Word approximants because otherwise it cannot be complete.

Our lower bound of $\omega * \omega$ for the symmetric short approximants $\approx^L$ does not quite match the upper bound of ω^ω provided by [8] and we conjecture that indeed, the exact convergence ordinal is $\omega * \omega$.

Finally, let us define the subclass of *decreasing* systems in the following way.

Definition 6.1 *A BPP description is* decreasing *if there is a linear order on variables such that for every rule $X \stackrel{a}{\longrightarrow} \alpha$ we have that α does not contain variables which are greater than X in chosen order.*

It seems that this subclass provides much structure to work with. Nevertheless, all systems presented in this paper are in fact decreasing. We believe that solving this class will be an important step towards a solution of the problem.

Acknowledgments. We are grateful to the reviewers for their constructive feedback on earlier drafts of this document and thank Colin Stirling for many helpful discussions.

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