On the Coverability Problem for Pushdown Vector Addition Systems in One Dimension

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Vector Addition Systems – Recap

Definition

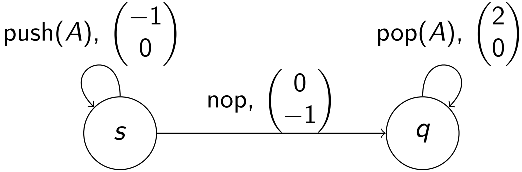
A VAS is a finite set of vectors $\mathbf{a} \in \mathbb{Z}^d$. For $\mathbf{v}, \mathbf{v}' : \mathbb{N}^d$ it has a step

$$\mathbf{v} \xrightarrow{a} \mathbf{v}' \quad \text{if} \quad \mathbf{v}' = \mathbf{v} + \mathbf{a}.$$

- Equivalent to Petri Nets
  (concurrency, weak counters, event systems)
- Reachability: decidable
  Mayr'81,*Kosaraju'82*, *. . . Leroux and Schmitz'15*
- Coverability, Boundedness: EXPSPACE-complete
  Lipton'76, *Rackoff'78*
- Most Games/Equivalences undecidable (e.g. Bisimulation)
  Jančar'95

Pushdown Vector Addition Systems

...are products of VAS with pushdown automata.



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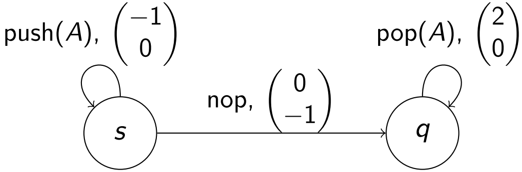

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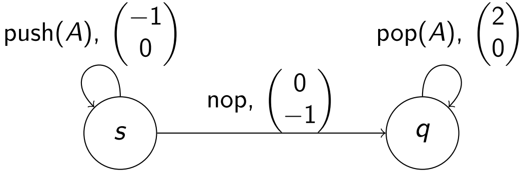

$$s, \bot, \begin{pmatrix}2\\1\end{pmatrix} \longrightarrow\longrightarrow s, AA\bot, \begin{pmatrix}0\\1\end{pmatrix}$$

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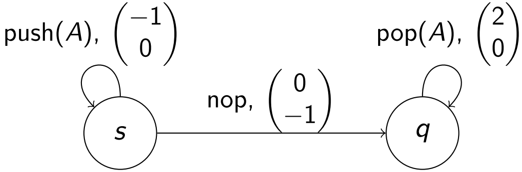

$$s, \bot, \begin{pmatrix}2\\1\end{pmatrix} \;\longrightarrow\longrightarrow\; s, AA\bot, \begin{pmatrix}0\\1\end{pmatrix} \;\longrightarrow\; q, AA\bot, \begin{pmatrix}0\\0\end{pmatrix}$$

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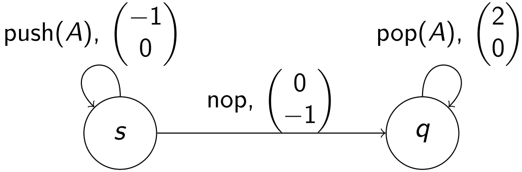

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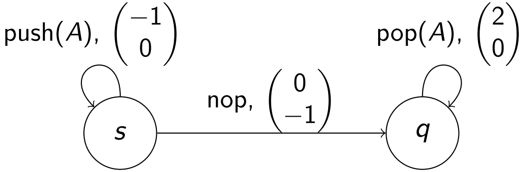

- Reachability = Coverability: decidability open
  TOWER-hard *Lazic'13*
- Boundedness: decidable with Hyper-Ackermannian bounds
  Leroux, Praveen, and Sutre'14

Pushdown Vector Addition Systems

Observation

Reachability in dim. d reduces to Coverability in dim. $d+1$.

$$\mathit{Reach}(0) \rightsquigarrow \mathit{Cover}(1) \rightsquigarrow \mathit{Reach}(1) \rightsquigarrow \mathit{Cover}(2) \rightsquigarrow \cdots$$

Our Contribution

Coverability for 1-dimensional Pushdown VAS is decidable.

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Another Perspective

Given:

- a VAS $\mathbf{A} \subset \mathbb{Z}^d$
- a context-free language $L \in \mathbf{A}^*$
- vectors $\mathbf{s}, \mathbf{t} \in \mathbb{N}^d$

Question: are there $\mathbf{a_1 a_2} \ldots \mathbf{a_k} \in L$ and $\mathbf{t'} \geq \mathbf{t}$ with

$$\mathbf{s} \xrightarrow{\mathbf{a_1}} \xrightarrow{\mathbf{a_2}} \cdots \xrightarrow{\mathbf{a_k}} \mathbf{t'} \quad ?$$

Grammar-Controlled Vector Addition Systems

Definition (GVAS)

A CfG $G = (V, A, R)$ where $A \subseteq \mathbb{Z}$.

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The *summary* of a nonterminal of $X \in V$ is

$$\textsc{Summary}_X(s) \stackrel{\text{def}}{=} \sup\{t \mid s \xrightarrow{X} t\}$$

Coverability:

$$\textsc{Summary}_X(s) \geq t\ ?$$

Example: Ackermann Functions $A_m : \mathbb{N} \to \mathbb{N}$

$$A_m(n) \stackrel{\text{def}}{=} \begin{cases} n+1 & \text{if } m = 0 \\ A_{m-1}^{n+1}(1) & \text{if } m > 0 \end{cases}$$

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$$A_m = \textsc{Summary}_{X_m}$$

$$\left(\text{via} \quad X_m \stackrel{*}{\Longrightarrow} -\mathbf{1}^n X_m X_{m-1}^n \Longrightarrow -\mathbf{1}^n \mathbf{1} X_{m-1}^{n+1} \stackrel{*}{\Longrightarrow} \cdots \right)$$

Flow Trees

Certificates for $\textsc{Summary}_S(c) \geq d$? Derivation trees!

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$\boxed{X_1}$

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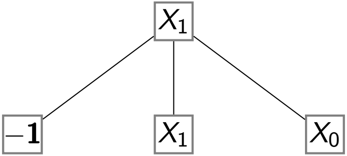

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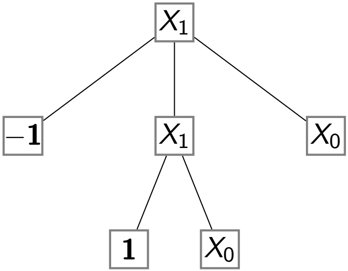

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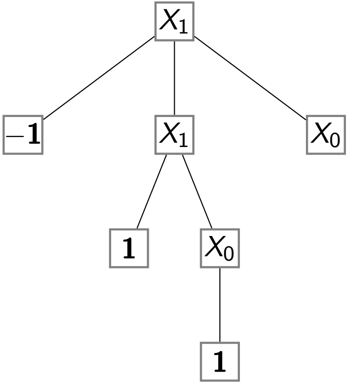

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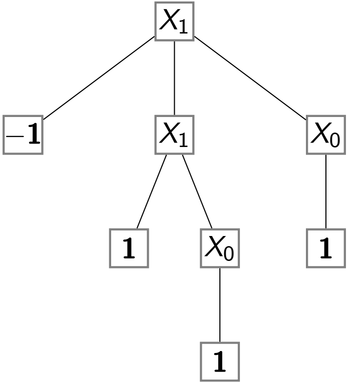

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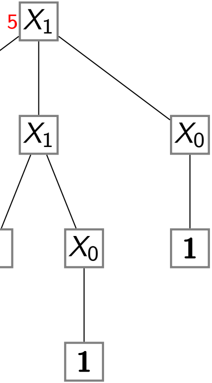

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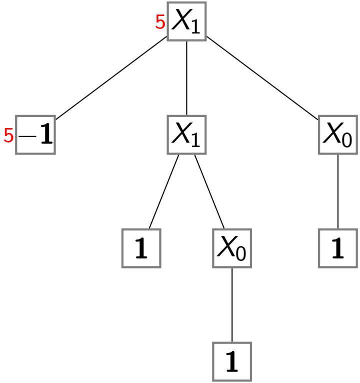

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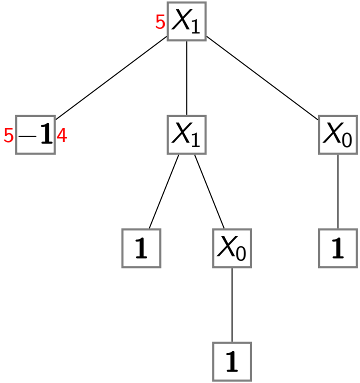

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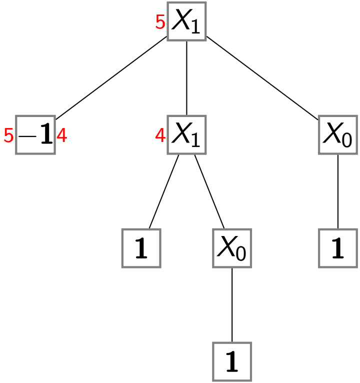

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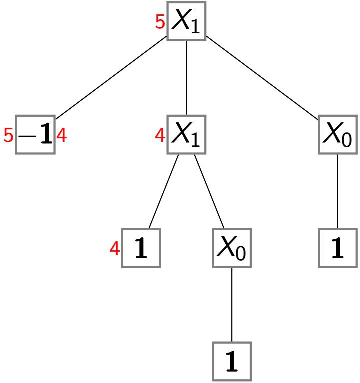

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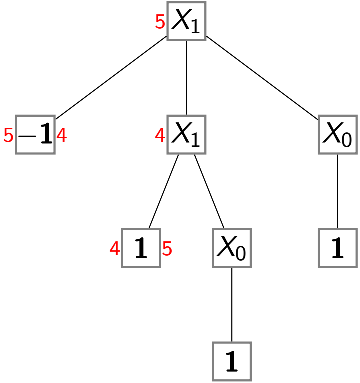

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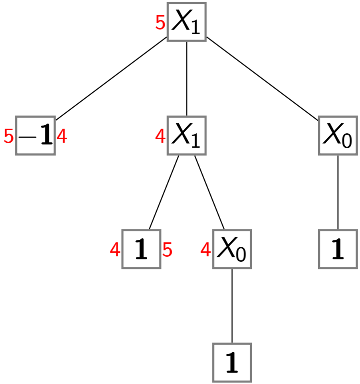

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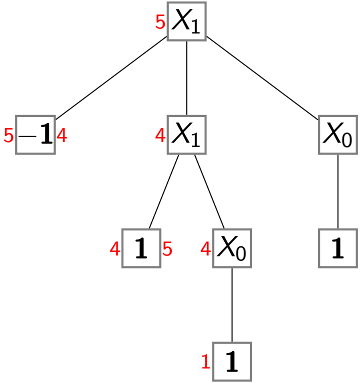

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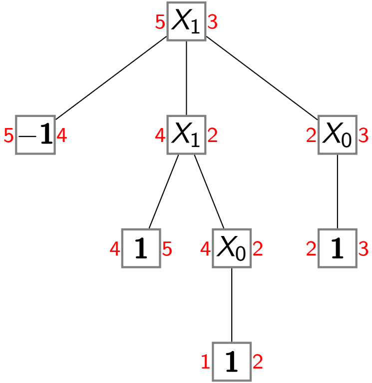

Flow Conditions

1. Nodes satisfy $\textsc{Summary}_X(IN) \geq OUT$
2. Labelling of neighbouring nodes is consistent

Flow Trees . . . can be arbitrarily large!

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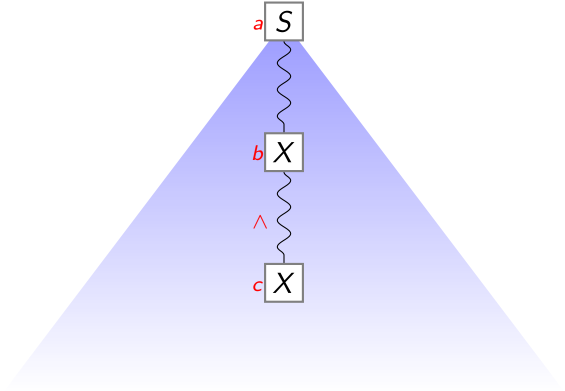
Pruning Flow Trees



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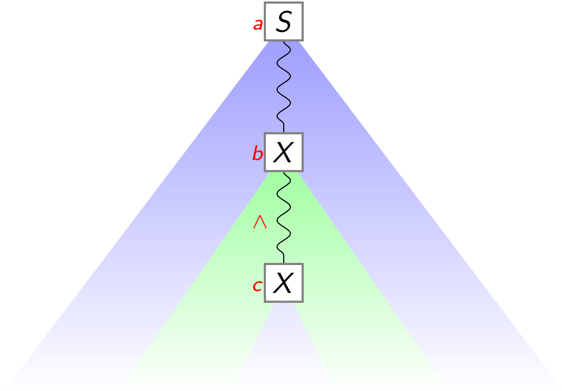

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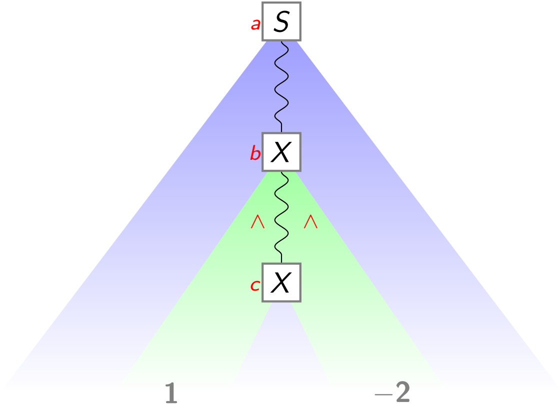

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Grammar for A_m

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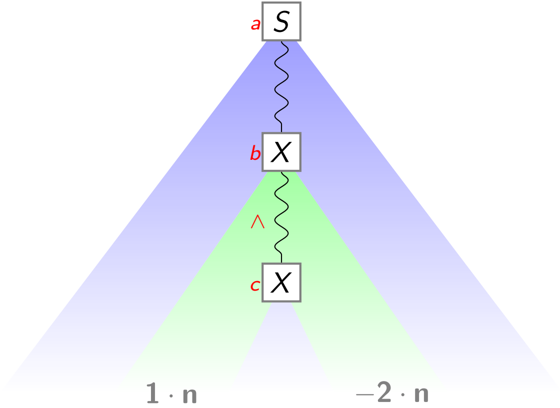

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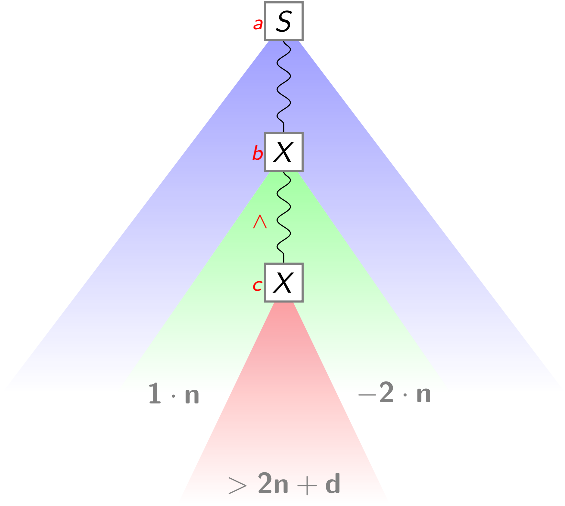

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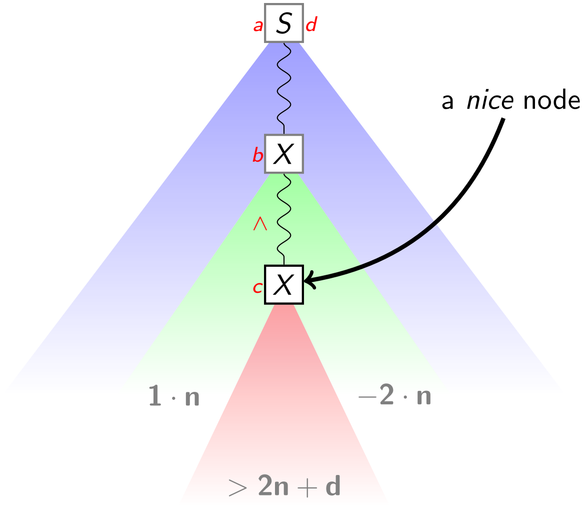

Main Argument for Decidability

Def.: A *Certificate*

is a flow tree where every leaf is either nice or has a finite ratio.

We show that

1. $c \xrightarrow{S} d' \geq d$ implies a small certificate with root $c\,\boxed{S}\,d$.
2. it is possible to check if $\textsc{Ratio}_X = \infty$
3. if $\textsc{Ratio}_X < \infty$, then $\textsc{Summary}_X$ is computable

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Conclusion

- Pushdown VAS coverability is decidable in dim. 1.
- Grammar-Controlled VAS, flow trees...
- Complexity (dim. 1)
  - Coverability: NP-hard; in EXPSPACE (?)
  - Boundedness: NP-hard; in EXPTIME.
- In arbitrary dimension d:
  - Reachability=Coverability: open
  - Boundedness: decidable; TOWER-hard; in $\mathbf{F}_{\omega^{\mathbf{d}}}$

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Questions?