The Inverse Characteristic Function in Argument Frameworks: Properties and Complexity

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Abstract

The characteristic function of a set of arguments S in a framework $\mathcal{H}$ is an important concept underpinning the formulation of most standard argumentation semantics, e.g. grounded, complete, admissible and strongly admissible. Within such frameworks, $\mathcal{F}_{\mathcal{H}}(S)$ (the characteristic function of S within $\mathcal{H}$) describes that set of arguments which S may be used to defend. In this work we define and consider the properties of an *inverse characteristic function*. This function, which we denote $\mathcal{F}_{\mathcal{H}}^{-1}$, given an argument y describes all subsets S for which $y \in \mathcal{F}_{\mathcal{H}}(S) \setminus S$. After reviewing some refinements of this idea, we show that any system of incomparable subsets $\mathbb{S}$ is such that a framework with $\mathcal{F}_{\mathcal{H}}^{-1}(y) = \mathbb{S}$ may be constructed. We further consider some natural decision problems associated with inverse characteristic functions and classify their complexity.

1. Introduction

The model of abstract argumentation promoted in the seminal article of Dung (1995) has focused attention on treatment of argumentation structures from a graph-theoretic perspective. This graph-theoretic view has proved to be of importance in complexity-theoretic terms e.g. as exemplified in work of Dimopoulos and Torres (1996), Dunne and Bench-Capon (2002), Dvořák and Woltran (2010), Dunne (2007, 2009). It also, however, provides the basis for set-theoretic treatments of argumentation semantics and labelling concepts so allowing novel notions of "collection of acceptable arguments" to be proposed. Among the many examples of such novel semantics we find semi-stable semantics from Caminada (2006), Caminada et al. (2012), the ideal semantics of Dung et al. (2007), and the formulation of strong admissibility given by Baroni and Giacomin (2007).

A central notion in defining argumentation semantics within Dung's formalism has been that of the *characteristic function.* We present a formal definition in Section 2, but for now note that the characteristic function $\mathcal{F}_{\mathcal{H}}(S)$ within a framework describes those arguments that can be defended by S. The interplay between $\mathcal{F}_{\mathcal{H}}(S)$ and S underpins semantics such as the grounded, complete, and admissible.

In this article our concern is not with the characteristic function itself but with a formulation of an *inverse function.* So, while $\mathcal{F}_{\mathcal{H}}(S)$ tells us what arguments S can defend, suppose we have some argument, y say, and wish to know what subsets, S, could be used to defend y, i.e. to determine which sets S are such that $y \in \mathcal{F}_{\mathcal{H}}(S)$. There are a few reasons why being able to manipulate such "inverse characteristic functions" may be useful. Such awareness may inform proof procedures in trying to demonstrate that y is acceptable under some semantics. For if we know that $y \in \mathcal{F}_{\mathcal{H}}(S)$ we can concentrate on demonstrating the acceptability of S in order to prove that y is acceptable. Thus an awareness of which subsets belong to those within the inverse characteristic function of y suggests an informal "back-tracking" procedure: in trying to establish y under some semantics, choose S with $y \in \mathcal{F}_{\mathcal{H}}(S)$ then try to establish the validity of arguments in S, i.e. for each $z \in S$ select some S_z for which $z \in \mathcal{F}_{\mathcal{H}}(S_z)$. Furthermore, faced with a number of alternatives $\{S_1, \ldots, S_r\}$ for which $y \in \mathcal{F}_{\mathcal{H}}(S_i)$ we can try to capture notions of "best" sets (e.g. as having smallest cardinality; or the smallest number of distinct attackers, etc).

In this paper, after presenting technical preliminaries in Section 2 together with divers formulations of the notion of inverse characteristic function, we then review properties of our formulation in Section 3.

In particular we show that given any incomparable system of subsets, $\mathbb{S}$ it is possible to construct a framework, $\mathcal{H}$, for which the inverse characteristic function of y contains precisely the sets in $\mathbb{S}$. We extend this construction completely to characterize those systems, $\mathbb{S}$, (whether incomparable or not) for which an AF having *exactly* the sets in $\mathbb{S}$ as the inverse characteristic function of a given argument can be built. An interesting side-effect of this characterization is rephrasing through a class of propositional logic functions. We denote this class by C_n. Not only are *monotone* Boolean functions, M_n, a strict subset of C_n there are, in addition, n-argument propositional functions not contained within it. We explore this class and its properties in greater depth within Section 4.

Some natural decision problems arising with inverse characteristic functions and their complexity are considered in Section 5 with conclusions reported in Section 6.

2. Preliminaries

We briefly restate some of the basic concepts in formal argumentation theory restricting to finite argumentation frameworks.

Definition 1. *An* argumentation framework *(*AF*) is a pair $\mathcal{H} = (\mathcal{X}, \mathcal{A})$ where $\mathcal{X}$ is a finite set of entities called* arguments *and $\mathcal{A}$ is a binary relation on $\mathcal{X}$. For any $p, q \in \mathcal{X}$ we say that p* attacks q *if $< p, q > \in \mathcal{A}$.*

Definition 2. *Let $\mathcal{H} = (\mathcal{X}, \mathcal{A})$ be an argumentation framework and x an argument in $\mathcal{X}$. We define $\{x\}^+$ to be the set of arguments that are attacked by x and $\{x\}^-$ as the set which attack x. The forms $\{x\}^+$ and $\{x\}^-$ are extended to* sets *of arguments S by defining S^+ as the union over all arguments $x \in S$ of $\{x\}^+$ with S^- defined in an analogous style. The notation $\nu(x)$ is used for $\{x\}^+ \cup \{x\}^-$. A subset S of $\mathcal{X}$ is said to be* conflict-free *if $S \cap S^+ = \emptyset$. A subset S is said to* defend x *if $\{x\}^- \subseteq S^+$. The* characteristic function *$\mathcal{F}_{\mathcal{H}} : 2^{\mathcal{X}} \to 2^{\mathcal{X}}$ is defined as $\mathcal{F}_{\mathcal{H}}(S) = \{x \mid S \text{ defends } x\}$.*

Definition 3. *Let $\mathcal{H} = (\mathcal{X}, \mathcal{A})$ be an argumentation framework. A subset S of the arguments is said to be:*

- *an admissible set if S is conflict-free and $S \subseteq \mathcal{F}_{\mathcal{H}}(S)$*
- *a complete extension if S is conflict-free and $S = \mathcal{F}_{\mathcal{H}}(S)$*
- *a grounded extension if S is the smallest (w.r.t. $\subseteq$) complete extension*

- *a preferred extension if S is a maximal (w.r.t. $\subseteq$) complete extension*
- *a strongly admissible set if S is admissible and for each $y \in S$, there is a strongly admissible subset T of $S \setminus \{y\}$ for which $y \in \mathcal{F}_{\mathcal{H}}(T)$.*

The notion of strong admissibility was originally presented by Baroni and Giacomin (2007). The structure given in Definition 3 is an equivalent formulated by Caminada (2014) and has been used in exploring computational and complexity issues relating to strong admissibility, e.g. Caminada and Dunne (2019b, 2019a).

We adopt the following notational conventions. Given an underlying set $\mathcal{X}$, we use upper case Roman letters, e.g. S, T, etc. for an arbitrary *subset* of $\mathcal{X}$, and lower case Roman letters, x, y, z, etc. for arbitrary *members* of $\mathcal{X}$. The notation $2^{\mathcal{X}}$ is used for the *set of all subsets* (sometimes referred to as the *powerset*) of $\mathcal{X}$, with $\mathbb{S}$, $\mathbb{T}$ etc. denoting subsets of $2^{\mathcal{X}}$. Thus, in this latter case, $\mathbb{S}$ is a *set* of subsets. We note the difference between $\mathbb{S} = \emptyset$ and $\mathbb{S} = \{\emptyset\}$ the former being the collection containing no sets whatsoever and the latter being the system whose only element is the empty set.

Semantics prescribe criteria to be satisfied by subsets of $\mathcal{X}$ in $\mathcal{H}(\mathcal{X}, \mathcal{A})$. We use

$$\sigma(\mathcal{H}) \quad = \quad \{\ S \subseteq \mathcal{X}\ :\ S \text{ satisfies the criteria described by } \sigma\ \}$$

Referring to $cf(\mathcal{H})$ and $adm(\mathcal{H})$ in the cases of conflict-free and admissible sets of $\mathcal{H}$.

The principal object of interest in the present paper is the idea of *inverse characteristic function.*

We start with the most general definition in Definition 4 and then look at how this can be refined.

Definition 4. *Given $y \in \mathcal{X}$ and $\mathcal{H}$ the* inverse characteristic function *of y with respect to $\mathcal{H}$ is denoted $\mathcal{F}_{\mathcal{H}}^{-1}(y)$ and consists of the subset of $2^{\mathcal{X} \setminus \{y\}}$ for which*

$$S \in \mathcal{F}_{\mathcal{H}}^{-1}(y) \quad \Leftrightarrow \quad y \in \mathcal{F}_{\mathcal{H}}(S) \setminus S$$

i.e. $\forall\ x \in \{y\}^{-} \quad x \in S^{+}$

Notice that Definition 4 places no restrictions on S. In this formulation, S is required to be neither admissible nor even conflict-free. When the characteristic function, $\mathcal{F}_{\mathcal{H}}$ is applied in practice, e.g. in specifying various semantic criteria such as admissible or complete sets, the domain of $\mathcal{F}_{\mathcal{H}}$ is often restricted to those S which are conflict-free. It is, therefore, sensible to limit $\mathcal{F}_{\mathcal{H}}^{-1}$ using some basic variants each of which can be tuned to individual semantic criteria.

Definition 5. *For $y \in \mathcal{X}$ and $\mathcal{H}$,*

a. $\mathcal{F}^{-1}_{\mathcal{H},\sigma}(y)$ *(called the* σ-inverse characteristic function*) is formed by the subset of* $2^{\mathcal{X}\setminus\{y\}}$ *for which*

$$S \in \mathcal{F}^{-1}_{\mathcal{H},\sigma}(y) \quad \Leftrightarrow \quad y \in \mathcal{F}_{\mathcal{H}}(S) \setminus S \text{ and } S \cup \{y\} \in \sigma(\mathcal{H})$$

b. $\digamma^{-1}_{\mathcal{H},\sigma}(y)$ *(the* minimal σ-inverse characteristic function*) has*

$$S \in \digamma^{-1}_{\mathcal{H},\sigma}(y) \quad \Leftrightarrow \quad S \in \mathcal{F}^{-1}_{\mathcal{H},\sigma}(y) \text{ and for every } T \subset S,\ T \notin \mathcal{F}^{-1}_{\mathcal{H},\sigma}(y)$$

c. $\mathbb{F}^{-1}_{\mathcal{H},\sigma}(y)$ *(the* maximal σ-inverse characteristic function*) has*

$$S \in \mathbb{F}^{-1}_{\mathcal{H},\sigma}(y) \quad \Leftrightarrow \quad S \in \mathcal{F}^{-1}_{\mathcal{H},\sigma}(y) \text{ and for every } T \supset S,\ T \notin \mathcal{F}^{-1}_{\mathcal{H},\sigma}(y)$$

It is, of course, easily seen that

$$\digamma^{-1}_{\mathcal{H},adm}(y) \quad \subseteq \quad \mathcal{F}^{-1}_{\mathcal{H},adm}(y) \quad \subseteq \quad \mathcal{F}^{-1}_{\mathcal{H},cf}(y) \quad \subseteq \quad \mathcal{F}^{-1}_{\mathcal{H}}(y)$$

We do not, necessarily, have $\digamma^{-1}_{\mathcal{H},adm}(y) \subseteq \digamma^{-1}_{\mathcal{H},cf}(y)$. Consider the system in Figure 1 with

$$\mathcal{X} = \{u, v, x, y, z\} \text{ and } \mathcal{A} = \{< v, z >, < z, x >, < x, u >, < u, y >\}.$$

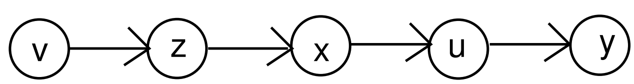

Figure 1: $\digamma^{-1}_{\mathcal{H},adm}(y) \not\subseteq \digamma^{-1}_{\mathcal{H},cf}(y)$.

In this $\{x\} \in \digamma^{-1}_{\mathcal{H},cf}(y)$: $y \in \mathcal{F}_{\mathcal{H}}(\{x\})$, $\{x, y\} \in cf(\mathcal{H})$ but the set $\{x, y\} \notin adm(\mathcal{H})$ since there is no defence to the attack on x from z, hence $\{x\} \notin \digamma^{-1}_{\mathcal{H},adm}(y)$. A minimal set in $\digamma^{-1}_{\mathcal{H},adm}(y)$ is $\{v, x\}$, however, although a member of $\mathcal{F}^{-1}_{\mathcal{H},cf}(y)$ it is not a minimal such element.

We further note the condition $S \cup \{y\} \in \sigma(\mathcal{H})$ instead of $S \in \sigma(\mathcal{H})$. For semantics such as conflict-freeness or strong admissibility

$$y \in \mathcal{F}_{\mathcal{H}}(S) \ \text{ and } \ S \cup \{y\} \in \sigma(\mathcal{H}) \quad \Rightarrow \quad S \in \sigma(\mathcal{H})$$

This, however, is not true of admissibility. For example, suppose we have a single argument, $\{z\}$ attacked by $\{w\}$. The set $\{z\}$ is not admissible. Now consider the case (still with $< w, z > \in \mathcal{A}$) where $\{< z, x >,\ < x, y >,\ < y, w >\}$ are in the set of attacks $\mathcal{A}$ (no other arguments or attacks being present). In this configuration $\{z\} \in \mathcal{F}^{-1}_{\mathcal{H},adm}(y)$: the set $\{z, y\}$ is admissible (although not strongly admissible) since y now provides a defence to the attack on z from w.

The implication

$$S \in \sigma(\mathcal{H}) \quad \text{and} \quad y \in \mathcal{F}_{\mathcal{H}}(S) \quad \Rightarrow \quad S \cup \{y\} \in \sigma(\mathcal{H})$$

will not always hold. Take the case $\sigma = cf$ and an AF in which $\{p, q\} \in cf(\mathcal{H})$, with no argument that attacks y and y attacking p. In this example $y \in \mathcal{F}_{\mathcal{H}}(\{p, q\})$ (y has no attackers and so is acceptable to every subset of arguments). The set $\{p, q, y\}$ is, however, not conflict-free (hence neither strongly admissible nor admissible).

A concept we will use particularly in Section 5 is that of the *standard translation* of a formula in CNF to an AF.

Definition 6. *Let $\varphi(x_1, \ldots, x_n)$ be a propositional formula in conjunctive normal form (*CNF*) having clauses $\{C_1, C_2, \ldots, C_m\}$ each C_j being a disjunction of* literals *(x_i or $\neg x_i$) drawn from $\{x_1, \ldots, x_n\}$.*

The standard translation *of φ to an* AF *is the framework $\mathcal{H}_\varphi$ with $2n+m+1$ arguments $\mathcal{X}_\varphi$ given by*

$$\mathcal{X}_\varphi \quad = \quad \{\ x_i, \neg x_i\ :\ 1 \leq i \leq n\ \}\ \cup\ \{\ C_j\ :\ 1 \leq j \leq m\ \}\ \cup\ \{\varphi\}$$

and attacks, $\mathcal{A}_\varphi$ formed as

$$\begin{array}{l} \{\ < x_i, \neg x_i >,\ \ < \neg x_i, x_i >\ \ :\ 1 \leq i \leq n\} \\ \{\ < x_i, C_j >\ \ :\ x_i \textit{ is a literal in } C_j\} \\ \{\ < \neg x_i, C_j >\ \ :\ \neg x_i \textit{ is a literal in } C_j\} \\ \{\ < C_j, \varphi >\ \ :\ 1 \leq j \leq m\} \end{array}$$

With some very minor variations the standard translation was introduced by Dimopoulos and Torres (1996) and used to demonstrate that deciding if a given argument belonged to any admissible set was NP–complete. Specifically, in the form given in Definition 6: there is an admissible set containing φ if and only if $\varphi(x_1, x_2, \ldots, x_n)$ is satisfiable.

Variants of the standard translation underpin many complexity-theoretic constructions in argumentation, e.g. Dunne and Bench-Capon (2002), Dvorak and Woltran (2010), Dunne (2007, 2009). A summary may be found in the article (Dunne & Wooldridge, 2009, Chapter 5).

3. Properties of the Inverse Characteristic Function

We first consider for which collections of subsets, $\mathbb{S}$, we can construct an AF, $\mathcal{H}$, with $\mathcal{F}^{-1}_{\mathcal{H},cf}(y) = \mathbb{S}$. It turns out that we can do so for any collection of incomparable sets. It is, of course, self-evident that should $\mathbb{S}$ be incomparable, that is for all $(S, T) \in \mathbb{S} \times \mathbb{S}$ if $S \subseteq T$ then $S = T$, then every $S \in \mathbb{S}$ is minimal with respect to $\subseteq$.

Before presenting this construction we first show that an "upward closure" condition must be met by any $\mathbb{S}$ satisfying $\mathbb{S} = \mathcal{F}^{-1}_{\mathcal{H}}(y)$. Formally

Lemma 1. *Given an* AF*, $\mathcal{H}(\mathcal{X}, \mathcal{A})$ and $y \in \mathcal{X}$ if $\mathbb{S} = \mathcal{F}^{-1}_{\mathcal{H}}(y)$ then for all $S \in \mathbb{S}$ and $T \subseteq \mathcal{X} \setminus \{y\}$, $S \cup T \in \mathbb{S}$.*

Proof. By definition, $S \in \mathcal{F}^{-1}_{\mathcal{H}}(y)$ if

$$\forall\ z \in \{y\}^-\ \ z \in S^+$$

We thus have from $z \in S^+$ that $z \in (S \cup T)^+$ for any $T \subseteq \mathcal{X} \setminus \{y\}$, hence $S \cup T \in \mathbb{S}$ if $\mathbb{S} = \mathcal{F}^{-1}_{\mathcal{H}}(y)$. $\square$

The property of Lemma 1 implies that $\mathbb{S}$, if describing $\mathcal{F}^{-1}_{\mathcal{H}}(y)$, is closed under $\cup$, should S and T be subsets within $\mathbb{S}$ then $S \cup T$ must also be in $\mathbb{S}$.

This upward closure property will not, in general be true of the cases $\mathcal{F}^{-1}_{\mathcal{H},cf}$ and $\mathcal{F}^{-1}_{\mathcal{H},adm}$. In fact, we do not necessarily have closure under union: S and T may be conflict-free, be such that $S^+ \supseteq \{y\}^-$ and $T^+ \supseteq \{y\}^-$ however $S \cup T$ is not conflict-free. For example in Figure 2, both $\{1\}$ and $\{2\}$ are members of $\mathcal{F}^{-1}_{\mathcal{H},cf}(3)$ however their union $\{1,2\}$ is not conflict-free (this set is in $\mathcal{F}^{-1}_{\mathcal{H}}(3)$, however).

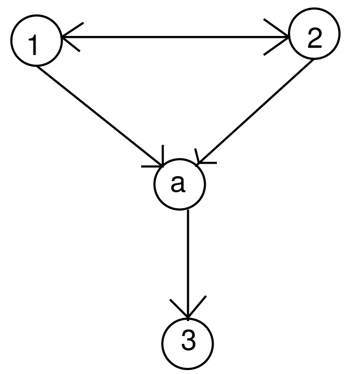

Figure 2: $\mathcal{F}^{-1}_{\mathcal{H},cf}(3) = \{\{1\},\{2\}\}$ and is not closed under $\cup$.

The liberal nature of the most general formulation of $\mathcal{F}^{-1}_{\mathcal{H}}$, specifically the upward closure requirement, and the absence of closure under $\cup$ within $\mathcal{F}^{-1}_{\mathcal{H},cf}$ provides another motivation for focusing attention on $\mathcal{F}^{-1}_{\mathcal{H},cf}(y)$, that is to say minimal conflict-free sets defending y in $\mathcal{H}$.

We can justify the term "inverse" through the fact that despite $\mathcal{F}_{\mathcal{H}}$ mapping from $2^{\mathcal{X}}$ to $2^{\mathcal{X}}$ and $\mathcal{F}^{-1}_{\mathcal{H}}$ from $\mathcal{X}$ to *subsets* of $2^{\mathcal{X}}$, focusing on the conflict-free variants of the latter there is an easily established link between the two.

Lemma 2. *For any* AF, $\mathcal{H}(\mathcal{X},\mathcal{A})$ *let* $y \in \mathcal{X}$.

a. For all $S \in \mathcal{F}^{-1}_{\mathcal{H},cf}(y)$, $y \in \mathcal{F}_{\mathcal{H}}(S)$.

b. If $S \cup \{y\}$ *is conflict-free,* $y \notin S$ *and* $y \in \mathcal{F}_{\mathcal{H}}(S)$ *then* $S \in \mathcal{F}^{-1}_{\mathcal{H},cf}(y)$.

Proof. Immediate from the definitions of $\mathcal{F}_{\mathcal{H}}$ and $\mathcal{F}^{-1}_{\mathcal{H},cf}(y)$. $\square$

We recall that a propositional function $f : \{\top,\bot\}^n \to \{\top,\bot\}$ with formal variables $X_n = \{x_1,\ldots,x_n\}$ is a *monotone* (increasing) function if for every $S \subseteq X_n$ for which $f[S] = \top$, any $T \supseteq S$ also has $f[T] = \top$. A well-known property of monotone propositional functions is that these have a unique minimal representation as a disjunction of products (DNF) equivalently as a conjunction of clauses (CNF). That is, if $f(X_n)$ is monotone there are unique systems of incomparable subsets $\{P_1,\ldots,P_r\}$ and $\{C_1,\ldots,C_t\}$ of 2^{X_n} for which

$$f(X_n) \;\equiv\; \bigvee_{i=1}^{r}\left(\bigwedge_{x_j\in P_i} x_j\right) \;\equiv\; \bigwedge_{i=1}^{t}\left(\bigvee_{x_j\in C_i} x_j\right)$$

Theorem 1. *Let $\mathbb{T} \subset 2^{\mathcal{X}\setminus\{y\}}$ be any system of incomparable subsets. and let $\mathcal{Z} = \cup_{T\in\mathbb{T}} T$. There is an* AF $\mathcal{H} = (\mathcal{X}, \mathcal{A})$ *in which $\{y\} \cup \mathcal{Z} \subseteq \mathcal{X}$ and $\mathbb{T} = F^{-1}_{\mathcal{H},cf}(y)$.*

Proof. Given $\mathbb{T}$ as in the Theorem statement consider the monotone propositional function $f_{\mathbb{T}}$ over the variables $\mathcal{Z} = (z_1, z_2, \ldots, z_n)$ defined via

$$f_{\mathbb{T}}(z_1, \ldots, z_n) \quad \equiv \quad \bigvee_{T\in\mathbb{T}} \left(\bigwedge_{z_i \in T} z_i \right)$$

It is clearly the case that $f_{\mathbb{T}}[S] = \top$ if and only if $S \supseteq T$ for some $T \in \mathbb{T}$.

This specification $f_{\mathbb{T}}$ is given in implicant form, We can translate $\mathbb{T}$ to another system of subsets over $\mathcal{Z}$,

$$\mathbb{P} \quad = \quad \{P_1,\ P_2,\ \ldots,\ P_m\}$$

and the sets in $\mathbb{P}$ are also incomparable with

$$f_{\mathbb{T}}(z_1, \ldots, z_n) \quad \equiv \quad \bigwedge_{k=1}^{m} \left(\bigvee_{z_i \in P_k} z_i \right)$$

Build the AF $\mathcal{H} = (\mathcal{X}, \mathcal{A})$ with $\mathcal{X} = \mathcal{Z} \cup \{y\} \cup \{p_1, p_2, \ldots, p_m\}$ m being the number of sets (i.e. clauses) in $\mathbb{P}$. Add attacks $< p_k, y >$ for each $1 \leq k \leq m$ and an attack $< z_i, p_j >$ whenever $z_i \in P_j \in \mathbb{P}$.

If $U \supseteq T \in \mathbb{T}$ then $U^+ = \{p_1, p_2, \ldots, p_m\}$ since the implicant (disjunction of product terms using $\mathbb{T}$) and implicate (conjunction of clauses using $\mathbb{P}$) describe exactly the same propositional function, i.e. $f_{\mathbb{T}}[U] = \top$.

The general construction is shown in Figure 3.

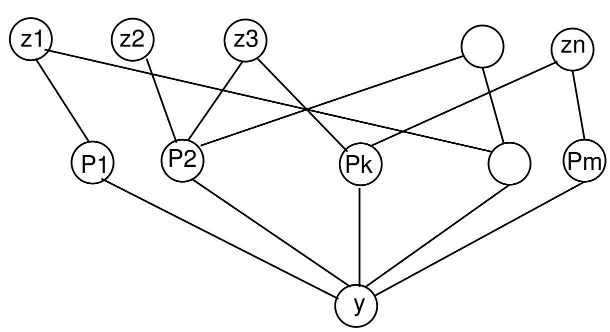

Figure 3: Realization of $F^{-1}_{\mathcal{H},cf}(y)$ as monotone CNF.

We have established that $F^{-1}_{\mathcal{H},cf}(y) \supseteq \mathbb{T}$, i.e. every set in $\mathbb{T}$ is a minimal conflict-free subset of $\mathcal{Z}$ defending y.

To complete the proof we need to show, in addition, that $F^{-1}_{\mathcal{H},cf}(y) \subseteq \mathbb{T}$.

Any $U \in F^{-1}_{\mathcal{H},cf}(y)$ can be written as

$$U \quad = \quad (u_1, u_2, \ldots, u_m) \qquad u_i \in P_i$$

[Notice that $(u_1, \ldots, u_m)$ is not necessarily a *set*: the same u_i may attack several distinct $p_j \in \{y\}^-$.]

We have,

$$f_{\mathbb{T}}(z_1, \ldots, z_n) \equiv \bigwedge_{k=1}^{m} \left(\bigvee_{z_i \in P_k} z_i \right)$$

and $f_{\mathbb{T}}[U] = \top$. However,

$$f_{\mathbb{T}}[U] \equiv \bigwedge_{k=1}^{m} u_i \equiv \bigvee_{T \in \mathbb{T}} \left(\bigwedge_{z_i \in T} z_i \right) [U]$$

Thus any $U \in F^{-1}_{\mathcal{H},cf}(y)$ is also in $\mathbb{T}$. □

This realization of an AF, $\mathcal{H}(\mathcal{X}, \mathcal{A})$ in which $F^{-1}_{\mathcal{H},cf}(y) = \mathbb{T}$ for some incomparable system of subsets $\mathbb{T}$ drawn from $\mathcal{Z}$ raises at least one question. Since it allows the set $\mathcal{Z}$ itself to be conflict-free it follows that any superset of $T \in \mathbb{T}$ is also contained in $\mathcal{F}^{-1}_{\mathcal{H},cf}$.

Suppose we wish to realize *exactly* the collection of incomparable sets specified by $\mathbb{T}$? In other words an AF with the property $F^{-1}_{\mathcal{H},cf}(y) = \mathcal{F}^{-1}_{\mathcal{H},cf}(y) = \mathbb{T}$. It turns out that this is straightforward to achieve.

Corollary 1. *Let $\mathbb{T} \subset 2^{\mathcal{X} \setminus \{y\}}$ be any system of incomparable subsets. and let $\mathcal{Z} = \cup_{T \in \mathbb{T}} T$. There is an* AF *$\mathcal{H} = (\mathcal{X}, \mathcal{A})$ in which $\{y\} \cup \mathcal{Z} \subseteq \mathcal{X}$, $F^{-1}_{\mathcal{H},cf}(y) = \mathcal{F}^{-1}_{\mathcal{H},cf}(y)$ and $\mathbb{T} = F^{-1}_{\mathcal{H},cf}(y)$.*

Proof. Recall the AF, $\mathcal{H} = (\mathcal{X}, \mathcal{A})$ which given $\mathbb{T} \subset 2^{\mathcal{Z}}$ simulates the propositional function,

$$f_{\mathbb{T}}(\mathcal{Z}) \equiv \bigvee_{T_r \in \mathbb{T}} \left(\bigwedge_{z_i \in T_r} z_i \right) \equiv \bigwedge_{P_k \in \mathbb{P}} \left(\bigvee_{z_i \in P_k} z_i \right)$$

As illustrated in Figure 3, this uses an argument, p_k, for each clause in $\mathbb{P}$. This argument p_k, in addition to attacking y, is attacked by those z_j for which $z_j \in P_k$.

We have seen that $F^{-1}_{\mathcal{H},cf}(y) = \mathbb{T}$ so in order to arrange $\mathcal{F}^{-1}_{\mathcal{H},cf}(y) = F^{-1}_{\mathcal{H},cf}(y)$ it suffices to eliminate all subsets, U of $\mathcal{Z}$ for which $U \supset T$ for some $T \in \mathbb{T}$.

Choose any such U letting $U = \{u_1, u_2, \ldots, u_t\}$. Since $U \supset T \in \mathbb{T}$ it must be the case that $U^+ = \{p_1, \ldots, p_m\}$ where $m = |\mathbb{P}|$. It is, however, also the case that $T^+ = \{p_1, \ldots, p_m\}$ (since $U \supset T$ and $T \in \mathbb{T}$).

Consider those arguments in $U \setminus T = \{v_1, \ldots, v_r\} \subset \mathcal{Z}$. For each $v_i \in U \setminus T$ we know the following:

a. $\{v_i\}^+ \subseteq \{p_1, \ldots, p_m\}$.

b. For each $p_j \in \{v_i\}^+$ there is some $z_k \in T$ having $p_j \in \{z_k\}^+$.

Property (b) following from the fact that $T \in F^{-1}_{\mathcal{H},cf}(y)$. From this property not only do we see $\{v_i, z_k\} \subseteq P_j$ but also that we do not need to have *both* present within a minimal conflict-free set defending y. It follows that we may modify $\mathcal{H}$ by adding all of the attacks

$$\{ \; < z_i, z_j >, \; < z_j, z_i > \; : \; \exists \; P_k \in \mathbb{P} \text{ with } \{z_i, z_j\} \subseteq P_k \; \}$$

In this configuration exactly one z_i can be chosen to attack a given clause (i.e. p_j) since using two or more distinct $z \in P_j$ will result in a non-conflict free set. We deduce that while the original sets $T \in \mathbb{T}$ will continue to satisfy $T \in F^{-1}_{\mathcal{H},cf}(y)$ from the fact that each such T will have $T^+ = \{p_1, \ldots, p_m\}$ no strict superset of T will be conflict-free, containing as it would, arguments z_i, z_j having $p_k \in \{z_i\}^+ \cap \{z_j\}^+$. □

As a basic example suppose we have the system of incomparable sets

$$\mathbb{S} \quad = \quad \{\{x_1\},\ \{x_2, x_3\},\ \{x_2, x_4, x_5\}\}$$

Then

$$f_{\mathbb{S}} \quad \equiv \quad x_1 \quad \vee \quad x_2x_3 \quad \vee \quad x_2x_4x_5$$

which in clausal form is

$$f_{\mathbb{P}} \quad \equiv \quad (x_1 \vee x_2)(x_1 \vee x_3 \vee x_4)(x_1 \vee x_3 \vee x_5)$$

To realise a system in which $F^{-1}_{\mathcal{H},cf}(y) = \mathbb{S}$ we have y attacked by three arguments $\{p_1, p_2, p_3\}$ corresponding to the three clauses of $f_{\mathbb{P}}$. These are, in turn, attacked by relevant subsets of $\{x_1, x_2, x_3, x_4, x_5\}$ so that $\{x_1\}^+ = \{p_1, p_2, p_3\}$; $\{x_2\}^+ = \{p_1\}$; $\{x_3\}^+ = \{p_2, p_3\}$; $\{x_4\}^+ = \{p_2\}$ and $\{x_5\}^+ = \{p_3\}$. In this configuration $F^{-1}_{\mathcal{H},cf}(y) \subset \mathcal{F}^{-1}_{\mathcal{H},cf}(y)$ (e.g. any strict superset of $\{x_1\}$ will be in $\mathcal{F}^{-1}_{\mathcal{H},cf}$). If we wish to ensure $F^{-1}_{\mathcal{H},cf}(y) = \mathcal{F}^{-1}_{\mathcal{H},cf}(y)$ we can do so, as indicated in the proof of Corollary 1, be adding symmetric attacks between x_1 and all of the arguments in $\{x_2, x_3, x_4, x_5\}$; and between x_3 and $\{x_4, x_5\}$. Now if T is a strict superset of one of $\{\{x_1\},\ \{x_2, x_3\},\ \{x_2, x_4, x_5\}\}$, e.g. $T = \{x_2, x_3, x_4\}$ then $T \notin \mathcal{F}^{-1}_{\mathcal{H},cf}(y)$ since (in this specific case) although $\{x_2, x_3, x_4\}^+ = \{p_1, p_2, p_3\}$ the subset $\{x_3, x_4\}$ is not conflict-free.

In Theorem 1 we showed that any system of incomparable sets, $\mathbb{S}$, (incomparability being equivalent to all members of $\mathbb{S}$ being minimal) may be realised via AFs, $\mathcal{H}$, in which $F^{-1}_{\mathcal{H},\sigma}(y) = \mathbb{S}$ for both $\sigma = cf$ and $\sigma = adm$. In this construction $F^{-1}_{\mathcal{H},\sigma}(y) \neq \mathcal{F}^{-1}_{\mathcal{H},\sigma}(y)$ as all supersets of $S \in \mathbb{S}$ belong to $\mathcal{F}^{-1}_{\mathcal{H},\sigma}(y)$.

In Corollary 1 we go to the other extreme in realizing $F^{-1}_{\mathcal{H},\sigma}(y)$ as the *only* sets in $\mathcal{F}^{-1}_{\mathcal{H},\sigma}(y)$.

There is, however, an intermediate possibility. What if we have a system, $\mathbb{S}$ which is not incomparable and wish to arrange $\mathcal{F}^{-1}_{\mathcal{H},\sigma}(y) = \mathbb{S}$? Is it the case that we can always do so or are there instances for which this is not possible?

As examples, using $\mathcal{Z} = \{z_1, z_2, z_3, z_4, z_5, z_6\}$ as our underlying set of arguments, we may wish to form $\mathcal{F}^{-1}_{\mathcal{H},\sigma}(y)$ to contain *exactly* the sets

1. $\{\{z_1\}, \{z_2, z_4\}, \{z_1, z_6\}, \{z_2, z_4, z_5\}, \{z_1, z_3, z_5, z_6\}\}$ or
2. $\{\{z_1\}, \{z_1, z_2\}, \{z_1, z_2, z_4\}, \{z_1, z_2, z_4, z_6\}, \{z_3, z_5\}\}$ or
3. $\{\{z_1\}, \{z_6\}, \{z_1, z_6\}, \{z_2\}, \{z_3\}, \{z_2, z_3\}, \{z_4, z_5, z_6\}\}$ etc.

In case (1) we have minimal sets $\{\{z_1\}, \{z_2, z_4\}\}$ but (among others) we do not have $\{\{z_1, z_3, z_5\}\}$. Similarly in case (2) we have minimal sets $\{\{z_1\}, \{z_3, z_5\}\}$ but do not all, e.g. $\{\{z_1, z_2, z_3, z_4, z_5, z_6\}\}$. Finally in case (3) the minimal sets are $\{\{z_1\}, \{z_2\}, \{z_3\}, \{z_6\}\}$ and the only allowable set involving z_4 or z_5 is $\{\{z_4, z_5, z_6\}\}$.

In examining these cases we have to consider not only the positive requirements of $S \in \mathbb{S} \cap \mathcal{F}^{-1}_{\mathcal{H},\sigma}(y)$ but also the constraints arising from more negative considerations: those of the form $S \notin \mathbb{S}$ therefore S should not be in $\mathcal{F}^{-1}_{\mathcal{H},\sigma}(y)$.

It is not too difficult to identify systems which cannot be realised as $\mathcal{F}^{-1}_{\mathcal{H},cf}(y)$.

Lemma 3. *Let $\mathcal{Z} = \{z_1, z_2, z_3\}$ and $\mathbb{S} = \{\{z_1\}, \{z_2\}, \{z_3\}, \{z_1, z_2, z_3\}\}$. There is no* AF*, $\mathcal{H}$ for which $\mathcal{F}^{-1}_{\mathcal{H},cf}(y) = \mathbb{S}$.*

Proof. Suppose the contrary holds and let $\mathcal{H}$ be a witnessing framework to $\mathcal{F}^{-1}_{\mathcal{H},cf}(y) = \{\{z_1\}, \{z_2\}, \{z_3\}, \{z_1, z_2, z_3\}\}$. From the fact that $\{z_1\}$ must belong to $\mathcal{F}^{-1}_{\mathcal{H},cf}(y)$ we must have $\{z_1\}^+ \supseteq \{y\}^-$ in $\mathcal{H}$. Similarly from $\{z_2\} \in \mathcal{F}^{-1}_{\mathcal{H},cf}(y)$ we need $\{z_2\}^+ \supseteq \{y\}^-$. Hence, $\{z_1, z_2\}^+ \supseteq \{y\}^-$ and since $\{z_1, z_2\} \notin \mathbb{S}$ we have, from our contradictory assumption, $\{z_1, z_2\} \notin \mathcal{F}^{-1}_{\mathcal{H},cf}(y)$ so that $\{z_1, z_2\} \notin cf(\mathcal{H})$. This, however, would imply $\{z_1, z_2, z_3\} \notin cf(\mathcal{H})$ and hence $\{z_1, z_2, z_3\} \notin \mathcal{F}^{-1}_{\mathcal{H},cf}(y)$, so contradicting $\mathcal{F}^{-1}_{\mathcal{H},cf}(y) = \mathbb{S}$. $\square$

Noting the construction from Figure 2, $|\mathcal{Z}| = 3$ is the least number of arguments for which constructions such as Lemma 3 can be applied. All of the systems $\{\emptyset, \{z_1\}, \{z_2\}, \{z_1, z_2\}\}$ in $2^{\{z_1,z_2\}}$ can be realised as $\mathcal{F}^{-1}_{\mathcal{H},adm}(y)$.

The special case illustrated in Lemma 3 provides the basis for a general nececessary condition that must hold in order for $\mathbb{S}$ to witness exactly the system of sets in $\mathcal{F}^{-1}_{\mathcal{H},cf}(y)$.

Definition 7. *Let $\mathbb{S} \subseteq 2^{\mathcal{Z}}$. The notation $\mu(\mathbb{S})$ describes the subset of $\mathbb{S}$ for which*

$$\mu(\mathbb{S}) \quad = \quad \{\ S \in \mathbb{S} \ \ : \ \ \forall\, T \in \mathbb{S} \ \ T \not\subset S\ \}$$

Thus $\mu(\mathbb{S})$ describes the minimal *sets in $\mathbb{S}$.*

Analogously, $\mathcal{M}(\mathbb{S})$ describes the maximal *sets in $\mathbb{S}$, that is*

$$\mathcal{M}(\mathbb{S}) \quad = \quad \{\ S \in \mathbb{S} \ \ : \ \ \forall\, T \in \mathbb{S}\ T \not\supset S\}$$

Let

$$\begin{array}{rcl} \lfloor S \rfloor & = & \{\ T \in \mu(\mathbb{S}) \ : \ T \subseteq S\} \\ \lceil S \rceil & = & \{\ T \in \mathcal{M}(\mathbb{S}) \ : \ S \subseteq T\} \end{array}$$

We say that $\mathbb{S}$ is closed with respect to subset intervals *if*

$$\forall U \in \mu(\mathbb{S}) \ \textit{and} \ \forall V \in \mathcal{M}(\mathbb{S}) : (U \subseteq T \subseteq V) \Rightarrow T \in \mathbb{S}$$

The systems $\mu(\mathbb{S})$ and $\mathcal{M}(\mathbb{S})$ are both systems of incomparable subsets of $\mathcal{Z}$. Furthermore if $\mathbb{S}$ is itself incomparable then, trivially, $\mu(\mathbb{S}) = \mathbb{S} = \mathcal{M}(\mathbb{S})$.

If we take an arbitrary $\mathbb{S} \subseteq 2^{\mathcal{Z}}$ and look at $S \in \mathbb{S}$ we have three possibilities.

S1. Exactly one of $S \in \mu(\mathbb{S})$ or $S \in \mathcal{M}(\mathbb{S})$ holds.

S2. $S \in \mu(\mathbb{S})$ *and* $S \in \mathcal{M}(\mathbb{S})$.

S3. S is in neither $\mu(\mathbb{S})$ nor $\mathcal{M}(\mathbb{S})$.

In the cases (S1) and (S3) we find sets U and V in $\mathbb{S}$ for which $U \subset S$ ($S \notin \mu(\mathbb{S})$) or $S \subset V$ ($S \notin \mathcal{M}(\mathbb{S})$). These sets are not necessarily unique. For example if

$$\mathbb{S} = \{\{z_1\}, \{z_2\}, \{z_5\}, \{z_1, z_2\}, \{z_1, z_2, z_3\}, \{z_1, z_2, z_4\}\}$$

Then $\mu(\mathbb{S}) = \{\{z_1\}, \{z_2\}, \{z_5\}\}$ and $\mathcal{M}(\mathbb{S}) = \{\{z_1, z_2, z_3\}, \{z_1, z_2, z_4\}, \{z_5\}\}$. The set $\{z_1, z_2\}$ has two strict supersets and two strict subsets which are also elements of $\mathbb{S}$. Finally the set $\{z_5\}$ is in $\mu(\mathbb{S}) \cap \mathcal{M}(\mathbb{S})$. This example is not closed with respect to subset intervals: $\{z_1\} \in \mu(\mathbb{S})$, $\{z_1, z_2, z_4\} \in \mathcal{M}(\mathbb{S})$ but $\{z_1, z_4\} \notin \mathbb{S}$. To arrange this property the sets

$$\{\{z_1, z_3\}, \{z_1, z_4\}, \{z_2, z_3\}, \{z_2, z_4\}\}$$

would have to be added to $\mathbb{S}$.

Theorem 2. *Let $\mathbb{S} \subseteq 2^{\mathcal{Z}}$. If there is an* AF*, $\mathcal{H} = (\mathcal{X}, \mathcal{A})$ having $\mathcal{Z} \cup \{y\} \subseteq \mathcal{X}$ in which $\mathbb{S} = \mathcal{F}^{-1}_{\mathcal{H},cf}(y)$ then $\mathbb{S}$ is closed with respect to subset intervals.*

Proof. Suppose $\mathbb{S} \subseteq 2^{\mathcal{Z}}$ and we have some AF, $\mathcal{H}$, for which $\mathbb{S} = \mathcal{F}^{-1}_{\mathcal{H},cf}(y)$. Assume, for the sake of contradiction, that $\mathbb{S}$ does not satisfy the closure property of the Theorem statement, allowing us to find three sets $U \in \mu(\mathbb{S})$, $V \in \mathcal{M}(\mathbb{S})$ and T for which $U \subseteq T \subseteq V$, $T \notin \mathbb{S}$. Since it is the case that $\mathcal{H}$ witnesses $\mathbb{S} = \mathcal{F}^{-1}_{\mathcal{H},cf}(y)$ we must have $U^+ \supseteq \{y\}^-$ and $V \cup \{y\} \in cf(\mathcal{H})$. From these we deduce $U \in cf(\mathcal{H})$, $T \in cf(\mathcal{H})$, and $T^+ \supseteq \{y\}^-$. The set T, from our assumption, is not in $\mathbb{S}$ and therefore not in $\mathcal{F}^{-1}_{\mathcal{H},cf}(y)$. From $U \subseteq T$ we know that $T^+ \supseteq \{y\}^-$; from $T \cup \{y\} \subseteq V \cup \{y\}$ we must have $T \cup \{y\} \in cf(\mathcal{H})$. The condition $T^+ \supseteq \{y\}^-$ indicates $y \in \mathcal{F}_{\mathcal{H}}(T)$ and we know the set $T \cup \{y\}$ to be conflict-free. These are exactly the conditions prescribed for a set to be in $\mathcal{F}^{-1}_{\mathcal{H},cf}(y)$ hence either $\mathcal{F}^{-1}_{\mathcal{H},cf}(y) \neq \mathbb{S}$ or we have $T \in \mathbb{S}$ in contradiction to our initial assumption. $\square$

We have seen, in Theorem 2, that closure with respect to subset intervals is a necessary condition for $\mathbb{S} = \mathcal{F}^{-1}_{\mathcal{H},cf}(y)$. We now show it also to be sufficient.

Theorem 3. *If $\mathbb{S} \subseteq 2^{\mathcal{Z}}$ is closed with respect to subset intervals then there is an* AF*, $\mathcal{H}$, for which $\mathbb{S} = \mathcal{F}^{-1}_{\mathcal{H},cf}(y)$.*

Proof. Let $\mathbb{S} \subseteq 2^{\mathcal{Z}}$ be closed with respect to subset intervals. From Theorem 1 we can construct $\mathcal{H}$ with arguments $\mathcal{X} = \mathcal{Z} \cup \{y\} \cup \{y\}^-$ for which $F^{-1}_{\mathcal{H},cf} = \mu(\mathbb{S})$. In addition, in this AF, $\{y\}^- = \{p_1, p_2, \ldots, p_m\}$ m being the number of clauses in the unique minimal CNF corresponding to $\mu(\mathbb{S})$. Let

$$\mathbb{P}_{\min} \;=\; \{P^{\min}_1,\ P^{\min}_2, \ldots,\ P^{\min}_m\} \quad P^{\min}_i \subseteq \mathcal{Z}$$

be the clauses of this minimal CNF so that

$$S \in \mathbb{S} \quad \Rightarrow \quad \left(\bigwedge_{i=1}^{m} \bigvee_{z_j \in P^{\min}_i} z_j \right) [S] \;\equiv\; \top$$

We saw in the construction from Theorem 1 that this admits any T which is a superset of $S \in \mu(\mathbb{S})$ as a set in $\mathcal{F}^{-1}_{\mathcal{H},cf}(y)$. In achieving $\mathcal{F}^{-1}_{\mathcal{H},cf}(y) = \mathbb{S}$ we need to add attacks between arguments in $\mathcal{Z}$ in such a way that if $T \supset U \in \mathcal{M}(\mathbb{S})$ then $T \notin cf(\mathcal{H})$. The construction of Corollary 1 ensured $\mathcal{F}^{-1}_{\mathcal{H},cf}(y) = \digamma^{-1}_{\mathcal{H},cf}(y) = \mu(\mathbb{S})$ by adding mutual attacks between any pair z_i and z_k occuring in the same clause of $P \in \mathbb{P}_{min}$. Consider now, however, the subset of $\mathbb{S}$ defined by $\mathcal{M}(\mathbb{S})$. This is again an incomparable system and, as we did with $\mu(\mathbb{S})$, there is a unique minimal CNF, with clauses

$$\mathbb{P}_{\max} \quad = \quad \{P_1^{\max},\ P_2^{\max}, \ldots,\ P_r^{\max}\} \quad P_i^{\max} \subseteq \mathcal{Z}$$

and for any $T \supseteq S$ with $S \in \mathcal{M}(\mathbb{S})$,

$$\left(\bigwedge_{i=1}^{r} \ \bigvee_{z_j \in P_i^{\max}} z_j \right) [T] \quad \equiv \quad \top$$

We can use those clauses in $\mathbb{P}_{\max}$ to determine which attacks should be added to $\mathcal{H}$ with $\digamma^{-1}_{\mathcal{H},cf}(y) = \mu(\mathbb{S})$. Consider the monotone propositional function, $f_{\mathbb{P}}^{\max}(\mathcal{Z})$ whose minimal CNF comprises exactly those clauses in $\mathbb{P}_{\max}$. If $U \subseteq \mathcal{Z}$ contains *exactly* one variable from each clause $P_i^{\max}$ of $\mathbb{P}_{\max}$ then $f_{\mathbb{P}}^{\max}[U] = \top$ and $U \in \mathcal{M}(\mathbb{S})$. If U contains more than one variable from some clause (while still having at least one variable from each) then $U \notin \mathcal{M}(\mathbb{S})$ since $U \notin \mathbb{S}$ (forming, as it does a strict superset of some maximal set). We can now use the properties of $\mathbb{P}_{\max}$ to determine which attacks to add to $\mathcal{H}$ realising $\mu(\mathbb{S}) = \digamma^{-1}_{\mathcal{H},cf}(y)$. We add attacks $\{< z_i, z_j >, < z_j, z_j >\}$ if there is some clause, $P_k^{\max}$ of $\mathbb{P}_{\max}$ that contains *both* z_i and z_j.

To see that the resulting AF, correctly realises $\mathbb{S} = \mathcal{F}^{-1}_{\mathcal{H},cf}(y)$ consider any $S \in \mathbb{S}$. Choosing any $T \in \lfloor S \rfloor$, $T \in \mu(\mathbb{S})$ so that $T \in \digamma^{-1}_{\mathcal{H},cf}(y)$. Choosing any $V \in \lceil S \rceil$ and $W \supset V$ will have $V \in \mathcal{F}^{-1}_{\mathcal{H},cf}(y)$ (V contains exactly one variable from each clause in $\mathbb{P}_{\max}$ and so no attacks have been added between the arguments in V). On the other hand $W \notin \mathcal{F}^{-1}_{\mathcal{H},cf}(y)$ as $W \notin cf(\mathcal{H})$: W contains at least two variables from some clause in $\mathbb{P}_{\max}$ and so mutual attacks between the corresponding arguments from $\mathcal{Z}$ have been added.

In summary the adjustments to $\mathcal{H}$ realising $\mu(\mathbb{S}) = \digamma^{-1}_{\mathcal{H},cf}(y)$ and the fact that $\mathbb{S}$ is closed with respect to subset intervals indicate that we can choose any $S \in \mathbb{S}$, identify a minimal subset, T, of this allowing $T^+ \supseteq \{y\}^-$ and $T \cup \{y\} \in cf(\mathcal{H})$. All of the supersets, U of T up to those belonging to $\mathcal{M}(\mathbb{S})$ will allow be conflict-free and continue to have $U^+ \supseteq \{y\}^-$. Once, however, we have formed a superset, V of T for which $V \notin \mathcal{M}(\mathbb{S})$ (so that $V \notin \mathbb{S}$) we will not have $V \in \mathcal{F}^{-1}_{\mathcal{H},cf}(y)$ since $V \notin cf(\mathcal{H})$. □

Returning to our earlier example,

$$\begin{array}{rcl} \mathbb{S} & = & \{\{z_1\},\{z_2\},\{z_5\},\{z_1,z_2\},\{z_1,z_2,z_3\},\{z_1,z_2,z_4\}\} \cup \\ & & \{\{z_1,z_3\},\{z_1,z_4\},\{z_2,z_3\},\{z_2,z_4\}\} \end{array}$$

with $\mu(\mathbb{S}) = \{\{z_1\},\{z_2\},\{z_5\}\}$ and $\mathcal{M}(\mathbb{S}) = \{\{z_1,z_2,z_3\},\{z_1,z_2,z_4\},\{z_5\}\}$ the whole system having the closed subset interval property.

$$\mathbb{P}_{\min} \quad = \quad \{\{z_1,z_2,z_5\}\} \quad ; \quad \mathbb{P}_{\max} \quad = \quad \{\{z_1,z_5\},\{z_2,z_5\},\{z_3,z_4,z_5\}\}$$

The AF realising $\digamma^{-1}_{\mathcal{H},cf}(y)$ as $\mu(\mathbb{S})$ would have a single attacker, p, of y with $\{p\}^- = \{z_1,z_2,z_5\}$. The arguments $\{z_3,z_4\}$ would have $\nu(z_3) = \nu(z_4) = \emptyset$. In order to eliminate supersets of sets in $\mathcal{M}(\mathbb{S})$ we add symmetric attacks between z_5 and all other arguments; and between z_3 and z_4. The former collection guarantees that no strict superset of $\{z_5\}$ is in $\mathcal{F}^{-1}_{\mathcal{H},cf}(y)$. In addition the set $\{z_2, z_3, z_4\} \notin \mathcal{F}^{-1}_{\mathcal{H},cf}(y)$. The only minimal set available is $\{z_2\}$ and the relevant maximal sets $\{\{z_1, z_2, z_3\}, \{z_1, z_2, z_4\}\}$, however $\{z_2, z_3, z_4\}$ is not a subset of any of these: the symmetric attack between z_3 and z_4 ensures that $\{z_2, z_3, z_4\} \notin cf(\mathcal{H})$ despite $\{z_2, z_3, z_4\}^+ \supseteq \{y\}^-$.

Noting that the constructions just presented apply equally to $\sigma = adm$, combining Theorem 2 and Theorem 3 we obtain Corollary 2.

Corollary 2. *For $\sigma \in \{adm, cf\}$, there is an* AF*, $\mathcal{H}$, in which $\mathbb{S} = \mathcal{F}^{-1}_{\mathcal{H},\sigma}(y)$ if and only if $\mathbb{S}$ is closed with respect to subset intervals.*

One other point raised by the the construction from Theorem 1 is that this may use exponentially many auxiliary arguments. This, of course, occurs cases for which $|\mathbb{T}| \sim 2^{|\mathcal{Z}|}$, e.g. when $|\mathcal{Z}| = 2n$ and $\mathbb{T}$ comprises all subsets of size n from $\mathcal{Z}$. Perhaps less obviously, one might have (again with $|\mathcal{Z}| = 2n$) $|\mathbb{T}| = n$ but, with the construction used, 2^n auxiliary arguments in $\mathcal{X}$. For example if

$$\mathbb{T} \ = \ \{ \ \{z_i, z_{n+i}\} \ : \ 1 \leq i \leq n \ \}$$

The implicant form of $f_{\mathbb{T}}(z_1, \ldots, z_{2n})$ used in the proof of Theorem 1 is

$$\bigvee_{i=1}^{n} \ z_i \wedge z_{n+i}$$

The implicate form giving rise to the system of sets $\mathbb{P}$ has 2^n clauses leading to $|\mathcal{X}| = 1 + n + 2^n$.

Such phenomena raise the question of whether this exponential increase is inevitable. Suppose we define for an incomparable system, $\mathbb{S}$, using only arguments drawn from a set $\mathcal{Z} = \{z_1, z_2, \ldots, z_n\}$ (by which it is assumed that for each $z_i \in \mathcal{Z}$ there is some $S \in \mathbb{S}$ with $z_i \in S$)

$$\min(\mathbb{S}) \ = \ \min \ \{ \ |\{y\}^-| \ : \ \exists \ \mathcal{H}(\mathcal{X}, \mathcal{A}), \ \{y\} \cup \mathcal{Z} \subseteq \mathcal{X}, \ F^{-1}_{\mathcal{H},cf}(y) = \mathbb{S}\}$$

Theorem 4. *For any incomparable system $\mathbb{S}$ from $2^{\mathcal{Z}}$ (in which $z_i \in \mathcal{Z}$ occurs in some $S \in \mathbb{S}$), let $\mathbb{P}_{\mathbb{S}}$ denote the collection of subsets from $\mathcal{Z}$ corresponding to the clauses in the unique minimal* CNF *expression equivalent to*

$$f_{\mathbb{S}}(\mathcal{Z}) \ \equiv \ \bigvee_{S \in \mathbb{S}} \ \bigwedge_{z \in S} \ z$$

So that

$$f_{\mathbb{S}}(\mathcal{Z}) \ \equiv \ \bigwedge_{P \in \mathbb{P}_{\mathbb{S}}} \left(\bigvee_{z \in P} z \right)$$

Let $cnf(\mathbb{S})$ be the number of clauses in this unique minimal CNF*.*

$$\min(\mathbb{S}) \ \geq \ cnf(\mathbb{S})$$

Proof. Given the system of incomparable sets, $\mathbb{S} \subset 2^{\mathcal{Z}}$ with $\mathcal{Z} = \{z_1, \ldots, z_m\}$ let $\mathcal{H}(\mathcal{X}, \mathcal{A})$ be such that $y \in \mathcal{X} \setminus \mathcal{Z}$, $\mathcal{Z} \subset \mathcal{X}$ and $\mathcal{F}^{-1}_{\mathcal{H},cf}(y) = \mathbb{S}$.

We first observe that in realizing $\mathbb{S}$ as $\mathcal{F}^{-1}_{\mathcal{H},cf}(y)$ the only arguments that are needed are those in $\mathcal{Z} \cup \{y\} \cup \{y\}^-$. To see this suppose we had some $x \notin \{y\}^{-1} \cup \mathcal{Z}$ in $\mathcal{X}$. We have a number of possibilities.

a. $\nu(x) \cap \mathcal{Z} \neq \emptyset$. We have some sub-cases.

  a1. Letting $S \subseteq \mathcal{Z}$ be such that $\nu(x) \cap S \neq \emptyset$. In this case $\{x\} \cup S \notin \mathcal{F}^{-1}_{\mathcal{H},cf}(y)$. Furthermore since $x \notin S$ for every $S \in \mathbb{S}$ it must also hold that $\{x\} \cup T \notin \mathcal{F}^{-1}_{\mathcal{H},cf}$ for every subset T of S. It follows x is redundant (with respect to realizing $\mathbb{S}$) as $\mathcal{F}^{-1}_{\mathcal{H},cf}$ and can be removed.

  a2. If $S \subseteq \mathcal{Z}$ with $\nu(x) \cap S = \emptyset$ then either $S^+ \cup \{x\}^+ \supseteq \{y\}^-$ or S is not conflict-free and hence $\{x\} \cup S \notin \mathcal{F}^{-1}_{\mathcal{H},cf}(y)$. In the former case some subset T of $\{x\} \cup S$ is in $\mathcal{F}^{-1}_{\mathcal{H},cf}(y)$ and we cannot have $x \in T$ since $T \in \mathbb{S}$. Again x can be removed without changing $\mathcal{F}^{-1}_{\mathcal{H},cf}(y)$.

From (a1), x cannot contribute to $\mathcal{F}^{-1}_{\mathcal{H},cf}(y)$ and from (a2) this continues to be so for those $S \subseteq \mathcal{Z}$ for which $\nu(x) \cap S = \emptyset$.

b. $\nu(x) \cap \{y\}^- \neq \emptyset$. From (a) we may deduce that $\nu(x) \cap \mathcal{Z} = \emptyset$. We are already working from the premise that $x \notin \{y\}^-$ and for this case to occur some $p \in \{y\}^-$ has $< p, x > \in \mathcal{A}$ or $< x, p > \in \mathcal{A}$. In both cases eliminating x from $\mathcal{H}$ will have no effect on $\mathcal{F}^{-1}_{\mathcal{H},cf}(y)$ being the system $\mathbb{S}$.

Hence any argument other than those in $\mathcal{Z} \cup \{y\}^-$ is redundant with respect to realising $\mathbb{S}$ as $\mathcal{F}^{-1}_{\mathcal{H},cf}(y)$

Consider $|\{y\}^-|$ in $\mathcal{H}$. Suppose for the sake of contradiction that $|\{y\}^-| < cnf(\mathbb{S})$.

We know from our initial analysis that $\{y\}^- \subseteq \mathcal{Z}^+$ and $\mathcal{X} = \mathcal{Z} \cup \{y\} \cup \{y\}^-$. Let

$$\{y\}^- \quad = \quad \{p_1, p_2, \ldots, p_r\}$$

with $r < cnf(\mathbb{S})$. Consider the system, $\mathbb{P} = \{P_1, P_2, \ldots, P_r\}$ of subsets of $\mathcal{Z}$ defined through $P_i = \{p_i\}^- \cap \mathcal{Z}$ and the propositional formula $f_{\mathbb{P}}(\mathcal{Z})$ equivalent to

$$f_{\mathbb{P}}(\mathcal{Z}) \quad = \quad \bigwedge_{j=1}^{r} \left(\bigvee_{z_i \in P_j} z_i \right)$$

We have assumed that $\mathbb{S} = \mathcal{F}^{-1}_{\mathcal{H},cf}(y)$ and so any $S \subset \mathcal{Z}$ for which $S \in \mathbb{S}$ must satisfy $f_{\mathbb{P}}[S] = \top$, i.e. contain at least representative from each P_j. There is, however, exactly one minimal CNF formula with the property that $S \in \mathbb{S}$ yields $f_{\mathbb{P}}[S] = \top$ so either the formula just constructed does not describe $\mathbb{S}$ as $\mathcal{F}^{-1}_{\mathcal{H},cf}$ or should it do so it has $r \geq cnf(\mathbb{S})$. $\square$

4. Propositional Functions and $\mathcal{F}^{-1}_{\mathcal{H},cf}(y)$

In this section our main interest is with questions arising from the following observation.

Given a set of n arguments, $\mathcal{X}$, and any argument $y \notin \mathcal{X}$, Corollary 2, characterizes exactly which subsets $\mathbb{S}$ of $2^{\mathcal{X}}$ allow frameworks, $\mathcal{H}$ to be built with $\mathbb{S} = \mathcal{F}^{-1}_{\mathcal{H},cf}(y)$. Given the correspondence between "subsets of $2^{\mathcal{X}}$" and "sets of Boolean assignments from $\{\top,\bot\}^{|\mathcal{X}|}$ to $\mathcal{X}$" via the mapping $\pi : 2^{\mathcal{X}} \to \{\top,\bot\}^{|\mathcal{X}|}$ defined for $S \subseteq \mathcal{X}$ as

$$\pi(x_i) \quad = \quad \begin{cases} \top & \text{if} \quad x_i \in S \\ \bot & \text{if} \quad x_i \notin S \end{cases}$$

we see that every $\mathbb{S} \subseteq 2^{\mathcal{X}}$ for which $\mathbb{S} = \mathcal{F}^{-1}_{\mathcal{H},cf}(y)$ has an associated propositional function $f_{\mathbb{S}} : \{\top,\bot\}^{|\mathcal{X}|} \to \{\top,\bot\}$ defined through

$$\forall\, U \subseteq \mathcal{X} \quad f_{\mathbb{S}}[U] = \top \quad \Leftrightarrow \quad U \in \mathbb{S}$$

In this section we wish to consider this class of propositional functions in more depth. We denote by C_n the class of propositional functions $f : \{\top,\bot\} \to \{\top,\bot\}$ such that

$$\forall\, \mathbb{S} \subseteq 2^{\mathcal{X}} \quad f_{\mathbb{S}}(x_1,\ldots,x_n) \in C_n \quad \Leftrightarrow \quad \exists \mathcal{H} \ s.t.\ \mathbb{S} = \mathcal{F}^{-1}_{\mathcal{H},cf}(y)$$

Recalling the standard notation, see e.g. Dunne (Dunne, 1988, pp. 7, 15), we use B_n for the set of all n argument propositional functions and M_n for the set of n argument *monotone* (increasing) propostional functions, i.e. for which

$$f(x_1,\ldots,x_n) \in M_n \quad \Leftrightarrow \quad \forall\, U,\ V \quad (f[U] = \top \ \text{ and } U \subseteq V) \ \Rightarrow \ f[V] = \top)$$

From the results presented in Section 3, specifically Corollary 2 we have

Theorem 5.

$$M_n \quad \subset \quad C_n \quad \subset \quad B_n$$

Proof. The containment $M_n \subseteq C_n$ is from Theorem 1 and Corollary 1. That the containment is strict is from Theorem 2 which also indicates $C_n \subset B_n$. □

We know, e.g. (Dunne, 1988, p. 122) that most (in fact, "almost all") propositional functions are not in M_n. Formally,

$$\lim_{n\to\infty} \frac{|M_n|}{|B_n|} \quad = \quad 0$$

Furthermore the total number of n argument propositional functions is easily shown to be 2^{2^n}. In contrast establishing *exact* bounds for $|M_n|$ is a classical open problem (Dedekind's Problem) for which the best estimates to date are those of (Korshunov, 1981).

In Theorem 6 the characterization of Corollary 2 is reworded.

Theorem 6. *For g, h in B_n we write $g \leq h$ whenever for all U, $g[U] \Rightarrow h[U]$.*

$$f(\mathcal{X}) \in C_{|\mathcal{X}|} \quad \Leftrightarrow \quad (f(\mathcal{X}) \equiv g(\mathcal{X}) \wedge (\neg h(\mathcal{X})),\ g,\ h \in M_n, \quad \textit{and} \quad h \leq g)$$

Proof. Suppose that $f(X) \equiv g(X_n) \wedge (\neg h(X_n))$ for monotone Boolean functions g and h with $h \leq g$. We show that

$$\mathbb{S}_f \quad = \quad \{ \ \{S\} \ : \ f[S] = \top \ \}$$

is closed with respect to subset intervals and therefore, via Corollary 2, belongs to C_n.

Let $\mathcal{M}_h(X_n)$ describe the maximal *falsifying* subsets of X_n with respect to h. That is to say, $\mathcal{M}_h$ is

$$\mathcal{M}_h \quad = \quad \{ \ \{U\} \ : \ h[U] = \bot \ \text{ and } \ (\forall \ W \supset U \quad h[W] = \top\}$$

Similarly, let $\mu_g(X_n)$ describe the minimal *satisfying* subsets of X_n with respect to g. That is,

$$\mu_g \quad = \quad \{ \ \{U\} \quad : \quad g[U] = \top \ \text{ and } \ (\forall \ V \subset U \quad g[V] = \bot\}$$

First notice that $\mu(\mathbb{S}_f) = \mu_g$ and $\mathcal{M}(\mathbb{S}_f) = \mathcal{M}_h$. To see this, consider any $S \in \mu(\mathbb{S}_f)$. By definition $g[S] = \top$ and $h[S] = \bot$ since $f_{\mathbb{S}}[S] = \top$. If it were the case some $V \subset S$ had $g[V] = \top$ then exactly the same subset would have $h[V] = \bot$ and hence $f_{\mathbb{S}}[V] = \top$ contradicting $S \in \mu(\mathbb{S}_f)$. Similarly from $S \in \mathcal{M}(\mathbb{S}_f)$ we have $g[S] = \top$ and $h[S] = \bot$ and were $W \supset S$ to be such that $h[W] = \bot$ then $g[W] = \top$ then, again, $f_{\mathbb{S}}[W] = \top$ contradicting $S \in \mathcal{M}(\mathbb{S}_f)$. To establish

$$(f(\mathcal{X}) \equiv g(\mathcal{X}) \wedge (\neg h(\mathcal{X})), \ g, \ h \in M_n, \ \text{ and } \ h \leq g) \quad \Rightarrow \quad f \in C_n$$

It suffices to show

$$\forall U \in \mu(\mathbb{S}_f) \text{ and } \forall V \in \mathcal{M}(\mathbb{S}_f) : (U \subseteq T \subseteq V) \Rightarrow T \in \mathbb{S}_f$$

From the definition of $\mathbb{S}_f$ it suffices to show

$$\forall U \in \mu(\mathbb{S}_f) \text{ and } \forall V \in \mathcal{M}(\mathbb{S}_f) : (U \subseteq T \subseteq V) \Rightarrow f[T] = \top$$

Consider any (U, V, T) with $U \in \mu(\mathbb{S}_f)$, $V \in \mathcal{M}(\mathbb{S}_f)$ and $U \subseteq T \subseteq V$. From $U \in \mu(\mathbb{S}_f)$ and $U \subseteq T$ we have $g[T] = \top$: $U \in \mu_g$, $g \in M_n$ and $U \subseteq T$. From $V \in \mathcal{M}(\mathbb{S}_f)$ and $T \subseteq V$ we have $h[T] = \bot$: $V \in \mathcal{M}_h$, $h \in M_n$ and $T \subseteq V$. Hence, from $f_{\mathbb{S}} \equiv g \wedge (\neg h)$ we obtain

$$f_{\mathbb{S}}[T] \quad = \quad g[T] \wedge (\neg h[T]) \quad = \quad \top \wedge (\neg \bot) \quad = \quad \top$$

So completing the first part.

It remains to show the converse implication

$$f(\mathcal{X}) \in C_{|\mathcal{X}|} \quad \Rightarrow \quad (f(\mathcal{X}) \equiv g(\mathcal{X}) \wedge (\neg h(\mathcal{X})), \ g, \ h \in M_n, \ \text{ and } \ h \leq g)$$

From $f(\mathcal{X}) \in C_{|\mathcal{X}|}$ and Corollary 2 the set $\mathbb{S}_f$, defined earlier, is closed with respect to subset intervals. Consider the propositional functions g and h defined as

$$g(x_1, \ldots, x_n) \quad \equiv \quad \bigvee_{S \in \mu(\mathbb{S}_f)} \ \bigwedge_{x_i \in S} x_i$$

$$h(x_1, \ldots, x_n) \quad \equiv \quad \bigvee_{S \in \mathcal{M}(\mathbb{S}_f)} \ \bigwedge_{x_i \in S} x_i \wedge \left(\bigvee_{x_j \notin S} x_j \right)$$

It is clear that $g \in M_n$ and $h \in M_n$.[1] We claim that $h \leq g$. To see this consider any $T \subseteq \{x_1, \ldots, x_n\}$ for which $h[T] = \top$. In this case we find some $V \in \mathcal{M}(\mathbb{S}_f)$ for which $V \subseteq T$ so that, from $h[V] = \top$ we have $h[T] = \top$. From the fact that there must be (at least) one $U \in \mu(\mathbb{S}_f)$ for which $U \subseteq V$ we have $g[U] = \top$ and, thence $g[T] = \top$ (as $U \subseteq V \subseteq T$). In total, $g \in M_n$, $h \in M_n$. amd $h \leq g$. To complete the argument we need to show

$$\forall \;\; S \in \mathbb{S}_f \quad g[S] = \top \quad \text{and } h[S] = \bot$$

Let T be any set in $\mathbb{S}_f$. Since it must be the case that $T \supseteq U$ for some $U \in \mu(\mathbb{S}_f)$ we must have $g[T] = \top$. It is, however, also the case that $T \subseteq V$ for some $V \in \mathcal{M}(\mathbb{S}_f)$. Examining the structure of $h(x_1, \ldots, x_n)$ in more detail, its implicants are all products (conjunctions) taking some $V \in \mathcal{M}(\mathbb{S}_f)$ and forming

$$\bigwedge_{x_i \in V} x_i \wedge \left(\bigvee_{x_j \notin V} x_j \right)$$

so that the only $W \subseteq \{x_1, \ldots, x_n\}$ for which $h[W] = \top$ are those which are a *strict superset* of a set in $\mathcal{M}(\mathbb{S}_f)$. For all other subsets, W, it holds that $h[W] = \bot$. We have chosen an arbitrary T in $\mathbb{S}_f$ and such T casnnot be a strict superset of any maximal set in $\mathbb{S}_f$, hence $h[T] = \bot$ and the required conclusion

$$\forall \;\; S \in \mathbb{S}_f \quad g[S] = \top \quad \text{and } h[S] = \bot$$

In total we deduce that

$$f(\mathcal{X}) \in C_{|\mathcal{X}|} \quad \Leftrightarrow \quad (f(\mathcal{X}) \equiv g(\mathcal{X}) \wedge (\neg h(\mathcal{X})),\; g,\; h \in M_n, \quad \text{and} \quad h \leq g)$$

as claimed. □

Corollary 3.

$$\lim_{n \to \infty} \frac{|C_n|}{|B_n|} \quad = \quad 0$$

That is to say, for almost all n-argument propositional functions, $f(x_1, \ldots, x_n)$ there is no AF *for which*

$$\{\; S \subseteq \{x_1, \ldots, x_n\} \;:\; f[S] = \top \;\} \quad = \quad \mathcal{F}^{-1}_{\mathcal{H},cf}(y)$$

Proof. From Theorem 6, every $f \in C_n$ is described by two monotone Boolean functions and hence $|C_n| \leq |M_n|^2$. Using Hansel's upper bound for $|M_n|$ from (Hansel, 1966) we have:[2]

$$|C_n| \leq \quad 3^{2\binom{n}{\lfloor n/2 \rfloor}}$$

1. Although we have not explicitly stated this, one of the defining attributes of M_n is that every (non-constant) $f \in M_n$ is equivalent to some propositional formula, φ_f built using the logical operations $\{\vee, \wedge\}$, cf. (Dunne, 1988, Lemma 1.1, p. 15).
2. Although the upper estimate by Hansel has been reduced, it suffices to use this for the proof, one advantage being the elegant function form obtained.

From Stirling's approximation we obtain

$$\binom{n}{\lfloor n/2 \rfloor} \quad \sim \quad \frac{2^n}{\sqrt{2\pi n}}$$

So that,

$$\log_2 |C_n| \quad = \quad O\left(\frac{2^n}{\sqrt{n}}\right)$$

On the other hand $\log_2 |B_n| = 2^n$ which being asymptotically larger than our estimate for $\log_2 |C_n|$ gives the result claimed. $\square$

5. Complexity Issues

Letting $\sigma \in \{adm, cf\}$ we have a number of computational issues motivated with respect to $\mathcal{F}^{-1}_{\mathcal{H},\sigma}(y)$ and $F^{-1}_{\mathcal{H},\sigma}(y)$. Specifically the following decision problems are raised.

<u>σ–VERIFICATION</u>

Instance: AF $\mathcal{H}(\mathcal{X}, \mathcal{A})$, argument $y \in \mathcal{X}$, $S \subseteq \mathcal{X} \setminus \{y\}$.

Question: $S \in \mathcal{F}^{-1}_{\mathcal{H},\sigma}(y)$?

<u>σ–NON-EMPTINESS</u>

Instance: AF $\mathcal{H}(\mathcal{X}, \mathcal{A})$, argument $y \in \mathcal{X}$

Question: $\mathcal{F}^{-1}_{\mathcal{H},\sigma}(y) \neq \emptyset$?

<u>σ–MINIMALITY</u>

Instance: AF $\mathcal{H}(\mathcal{X}, \mathcal{A})$ argument $y \in \mathcal{X}$, $S \subseteq \mathcal{X} \setminus \{y\}$.

Question: $S \in F^{-1}_{\mathcal{H},\sigma}(y)$?

<u>σ–MAXIMALITY</u>

Instance: AF $\mathcal{H}(\mathcal{X}, \mathcal{A})$ argument $y \in \mathcal{X}$, $S \subseteq \mathcal{X} \setminus \{y\}$.

Question: $S \in \mathbb{F}^{-1}_{\mathcal{H},\sigma}(y)$?

<u>σ–COINCIDENCE</u>

Instance: AF $\mathcal{H}(\mathcal{X}, \mathcal{A})$ argument $y \in \mathcal{X}$

Question: $F^{-1}_{\mathcal{H},\sigma}(y) = \mathcal{F}^{-1}_{\mathcal{H},\sigma}(y)$?

Two of these problems are easily classified.

Theorem 7.

a. σ–VERIFICATION *is in* P *for both* $\sigma = adm$ *and* $\sigma = cf$.

b. σ–NON-EMPTINESS *is* NP*–complete for both* $\sigma = adm$ *and* $\sigma = cf$.

Proof. For (a) verifying that $S \in \mathcal{F}^{-1}_{\mathcal{H},\sigma}(y)$ requires only checking $S \cup \{y\} \in \sigma(\mathcal{H})$ and that $y \in \mathcal{F}_{\mathcal{H}}(S)$ both of which are efficiently decidable.

For (b) simply use the standard translation of Definition 6 from a CNF-SAT of CNF formula $\varphi(x_1, \ldots, x_n)$ having clauses $\{C_1, \ldots, C_m\}$ to an AF, $\mathcal{H}_\varphi(\mathcal{X}_\varphi, \mathcal{A}_\phi)$. The instance $(\mathcal{H}, \varphi)$ of σ–NON-EMPTINESS is accepted, i.e. $\mathcal{F}^{-1}_{\mathcal{H}_\varphi,\sigma}(\varphi) \neq \emptyset$ if and only if $\varphi(x_1, \ldots, x_n)$ is satisfiable. □

With a little more effort we can also show

Theorem 8. cf-MINIMALITY *and* cf-MAXIMALITY *are in* P.

Proof. Given $\mathcal{H}(\mathcal{X}, \mathcal{A})$, $y \in \mathcal{X}$ and $S \subseteq \mathcal{X}$, in order to check $S \in \mathcal{F}^{-1}_{\mathcal{H},cf}$ we first need to confirm both $S \cup \{y\} \in cf(\mathcal{H})$ and $y \in \mathcal{F}_{\mathcal{H}}(S)$: if either fails to hold then $S \notin \mathcal{F}^{-1}_{\mathcal{H},cf}$ since $S \notin \mathcal{F}^{-1}_{\mathcal{H},cf}$. Both of these preconditions are verifiable efficiently. Suppose we have determined that $S \in \mathcal{F}^{-1}_{\mathcal{H},cf}(y)$. If S is not minimal then there is some $T \subset S$ for which $T \in \mathcal{F}^{-1}_{\mathcal{H},cf}(y)$. For each $x \in S$ let S_x be $S \setminus \{x\}$. We can first check for each x in turn whether $S_x^+ \supseteq \{y\}^-$. If for every $x \in S$ it turns out that $S_x^+ \not\supseteq \{y\}^-$ then $S \in \mathcal{F}^{-1}_{\mathcal{H},cf}(y)$ since it is not possible to remove any argument from S and preserve a defense of y. On the other hand if we find a single $x \in S$ for which $S_x^+ \supseteq \{y\}^-$ then $S_x \in \mathcal{F}^{-1}_{\mathcal{H},cf}(y)$ and $S_x \subset S$ thence $S \notin \mathcal{F}^{-1}_{\mathcal{H},cf}$.

For checking cf-maximality it suffices first to verify that $S \in \mathcal{F}^{-1}_{\mathcal{H},cf}(y)$ and then confirm for all $z \in \mathcal{X} \setminus (S \cup \{y\})$ that $S \cup \{z, y\} \notin cf(\mathcal{H})$ (notice that $S \in \mathcal{F}^{-1}_{\mathcal{H},cf}(y)$ immediately gives $S^+ \cup \{z\}^+ \supseteq \{y\}^+$ so in order for S to be maximal all supersets must be shown not to be conflict-free). □

Notice that the argument used in this proof does not extend to *adm*-minimality. In the proof we exploit the fact that every subset of a conflict-free set is conflict-free, however, it is not the case that every subset of an admissible set is admissible.

We also observe that the reduction used in the proof that σ–NON-EMPTINESS is NP–complete can not be applied to establish σ–COINCIDENCE is coNP–complete. If we use the standard translation, attempting to reduce from CNF–unsatisfiability to σ–COINCIDENCE then, although unsatisfiability does imply coincidence (since unsatisfiability implies $\mathcal{F}^{-1}_{\mathcal{H},\sigma}(\varphi) = \emptyset$ and so, vacuously, $\mathcal{F}^{-1}_{\mathcal{H},\sigma}(\varphi) = \mathcal{F}^{-1}_{\mathcal{H},\sigma}(\varphi)$), the converse – $\mathcal{F}^{-1}_{\mathcal{H},\sigma}(\varphi) = \mathcal{F}^{-1}_{\mathcal{H},\sigma}(\varphi)$ implies φ is unsatisfiable – is not, necessarily true: for example φ could have a *unique* satisfying assignment or only satisfying assignments which require *every* variable to be assigned some truth value, i.e. φ has no redundant literals.

Nevertheless, in the case *adm*-COINCIDENCE we are able to prove:

Theorem 9. *adm*–COINCIDENCE *is co*NP*–complete.*

Proof. Given the instance $(\mathcal{H}(\mathcal{X},\mathcal{A}),y)$ checking if $\digamma^{-1}_{\mathcal{H},adm}(y) = \mathcal{F}^{-1}_{\mathcal{H},adm}(y)$ just requires testing if

$$\forall S \subseteq \mathcal{X},\ \forall T \subseteq \mathcal{X} \quad (S \subset T) \Rightarrow (S \notin \mathcal{F}^{-1}_{\mathcal{H},adm}(y) \text{ or } T \notin \mathcal{F}^{-1}_{\mathcal{H},adm}(y))$$

Since testing membership in $\mathcal{F}^{-1}_{\mathcal{H},adm}(y)$ can be done efficiently this computation can be realised in coNP. It is noted that $S \subset T$ and both $S \in \mathcal{F}^{-1}_{\mathcal{H},adm}(y)$ and $T \in \mathcal{F}^{-1}_{\mathcal{H},adm}(y)$ guarantees $\digamma^{-1}_{\mathcal{H},adm}(y) \neq \mathcal{F}^{-1}_{\mathcal{H},adm}(y)$ even though it may not be the case that $S \in \digamma^{-1}_{\mathcal{H},adm}(y)$.

To show *adm*-COINCIDENCE is coNP–hard we use a variant of the standard translation from propositional formulae in CNF arguing that $\varphi(x_1,\ldots,x_n)$ is unsatisfiable if and only the constructed AF, $\mathcal{G}_\varphi$ has $\digamma^{-1}_{\mathcal{G},adm}(\psi) = \mathcal{F}^{-1}_{\mathcal{G},adm}(\psi)$

Given $\varphi(x_1,\ldots,x_n)$ a propositional formula in CNF modify $\mathcal{H}_\varphi(\mathcal{X}_\varphi,\mathcal{A}_\varphi)$ of the standard translation to the AF, $\mathcal{G}$, by adding arguments

$$\{\ \alpha,\ \vartheta,\ \pi,\ \psi\}$$

and attacks

$$\left\{ \begin{array}{l} <\varphi,\alpha>,\ \ <\varphi,\pi>,\ \ <\vartheta,\pi>,\ \ <\pi,\psi>, \\ <\alpha,x_i>,\ \ <\alpha,\neg x_{>}\ :\ 1 \leq i \leq n \end{array} \right\}$$

The construction is illustrated in Figure 4.

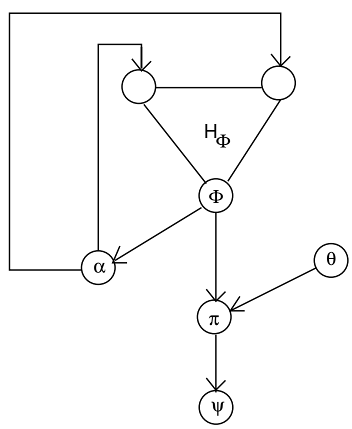

Figure 4: Variant of Standard Translation in Reduction to *adm*–COINCIDENCE.

We claim that $(\mathcal{G},\psi)$ is accepted as an instance of *adm*-COINCIDENCE if and only if $\varphi(x_1,\ldots,x_n)$ is unsatisfiable.

Suppose that $(\mathcal{G},\psi)$ is accepted as an instance of *adm*-COINCIDENCE. Notice that $\mathcal{F}^{-1}_{\mathcal{G},adm}(\psi)$ contains the set $\{\{\vartheta\}\}$. The argument ϑ is one attacker of π the only attacker of ψ. It follows that $\{\vartheta,\psi\}$ is admissible and since $\{\vartheta\}^- = \emptyset$ (hence $\{\vartheta\}$ is admissible) so $\psi \in \mathcal{F}_\mathcal{G}(\{\vartheta\})$. It is clear that $\{\{\vartheta\}\} \in \digamma^{-1}_{\mathcal{G},adm}(\psi)$. Suppose some other S was a member of $\mathcal{F}^{-1}_{\mathcal{G},adm}(\psi)$. If $\vartheta \in S$ and $S \neq \{\vartheta\}$ the assumption that $(\mathcal{G},\psi)$ is accepted as an instance of *adm*-COINCIDENCE would be contradicted: $S \notin \digamma_{\mathcal{G},adm}(\psi)$ (since $\vartheta \in S$). Since we cannot have $\vartheta \in S$ in order for $S \in \mathcal{F}^{-1}_{\mathcal{G},adm}(\psi)$ requires $\varphi \in S$. Such would require a subset $\{y_1,\ldots,y_n\}$ of the arguments $\{x_i,\ \neg x_i\ :\ 1 \leq i \leq n\}$ in order to attack all of the clause arguments C_j. Such a set would, however, indicate that φ was satisfiable. This allows us to deduce that if $(\mathcal{G}, \psi)$ is accepted as an instance of *adm*-COINCIDENCE the only set in $F^{-1}_{\mathcal{G},adm}(\psi)$ is $\{\{\vartheta\}\}$ and there are no other sets in $\mathcal{F}^{-1}_{\mathcal{G},adm}(\psi)$. Notice that since the argument α attacks each x_i and $\neg x_i$ and α is only counterattacked by φ, although, given $\underline{y} \in \{0,1\}^n$ all sets of the form

$$S_{\underline{y}} \quad = \quad \{\vartheta\} \ \cup \ \{x_i \ : \ y_i = 1\} \ \cup \ \{\neg x_i \ : \ y_i = 0\}$$

belong to $\mathcal{F}^{-1}_{\mathcal{G},cf}(\psi)$ these do not belong to $\mathcal{F}^{-1}_{\mathcal{G},adm}(\psi)$ since they do not contain a defence to the attacks by α.

We deduce that if $(\mathcal{G}, \psi)$ is accepted as an instance of *adm*-COINCIDENCE then $\varphi(x_1, \ldots, x_n)$ is unsatisfiable.

For the converse implication suppose that $\varphi(x_1, \ldots, x_n)$ is unsatisfiable. We show in this event that $(\mathcal{G}, \psi)$ is accepted as an instance of *adm*-COINCIDENCE. Since φ is unsatisfiable not only is there no admissible subset of the arguments in $\mathcal{G}$ that contains φ there is, furthermore, no admissible set that contains any of the arguments x_i or $\neg x_i$. In order for the latter arguments to be admissible a witnessing set would have to contain a defence to the attack from α, however, the only such defender would be φ which is itself inadmissible. It follows that the only subset, S, for which $\psi \in \mathcal{F}_{\mathcal{G}}(S)$ is the set $\{\vartheta\}$, hence $\mathcal{F}^{-1}_{\mathcal{G},adm}(\psi) = \{\{\vartheta\}\}$ and $\mathcal{F}^{-1}_{\mathcal{G},adm}(\psi) = F^{-1}_{\mathcal{G},adm}(\psi)$. It follows that if $\varphi(x_1, \ldots, x_n)$ is unsatisfiable then $(\mathcal{G}, \psi)$ is accepted as an instance of *adm*-COINCIDENCE so completing the proof that *adm*-COINCIDENCE is coNP–complete. $\square$

We have as an immediate Corollary of Theorem 9.

Corollary 4. *adm*-MAXIMALITY *is co*NP*–complete.*

Proof. For membership in coNP given $(\mathcal{H}, y, S)$ it suffices to test $S \in \mathcal{F}^{-1}_{\mathcal{H},adm}(y)$ and for every $T \supset S$ that $T \cup \{y\} \notin adm(\mathcal{H})$.

For coNP–hardness we use exactly the same translation from CNF formulae, φ to the AF, $\mathcal{G}$, described in the proof of Theorem 9. The instance of *adm*-MAXIMALITY is $(\mathcal{G}, \psi, \{\vartheta\})$. We have already seen that $\{\vartheta\} \in \mathcal{F}^{-1}_{\mathcal{G},adm}(\psi)$. It will be a maximal such set if and only if no other arguments of $\mathcal{G}$ can be added and admissibility preserved. Following the argument of Theorem 9 this will be the case if and only φ is unsatisfiable. $\square$

6. Conclusions & Further work

The principal aim of this paper has been to propose a formulation of the concept "inverse characteristic function" as a complementary notion to the well-studied standard idea of characteristic function. Our main efforts have been directed towards considering different plausible formulation from entirely unrestricted through to cases imposing some semantic constraint on the subsets of arguments allowed. We have argued that, in keeping with its prevalence in formulating semantics in Dung's schema, the most reasonable of these formulations is to require $S \cup \{y\}$ to be conflict-free if S is to be considered as a candidate inverse for y. After reviewing ideas of minimal and maximal sets within the formalism, we proceeded completely to characterize which subsets of a set $\mathcal{X}$ describe possible inverses later rephrasing this characterization in terms of a specific class of propositional functions. Finally we formulated some natural decision problems within the model classifying complexity status for all but two of these. The two unclassified cases – *adm*–MINIMALITY and *cf*–COINCIDENCE – are the focus of on-going work, however while it seems plausible to to conjecture that *adm*–MINIMALITY is coNP–complete (membership in coNP being straightforward), the status of *cf*–COINCIDENCE is less clear. Finally, in additions to exploring how these concepts can be exploited within proof procedures, the result presented in Theorem 4 giving exact bounds on the number of auxiliary arguments required to bring about a specific behaviour, raises a number of questions of interest. Among such would be examining related questions (bounds on numbers of arguments) in contexts other than that which has been the main focus of the current paper.

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