D-Separation

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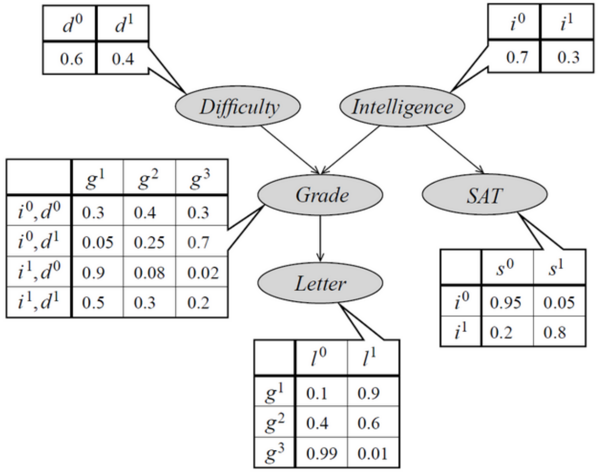
Up to now,

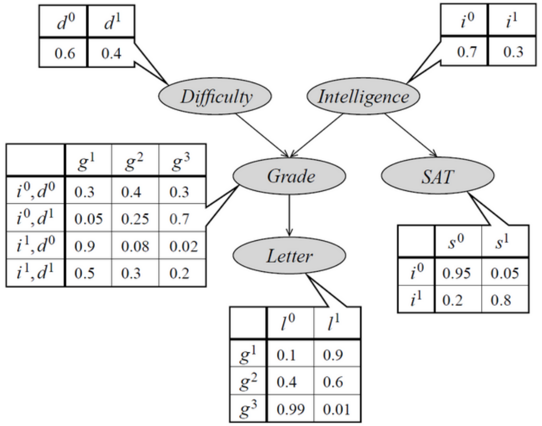
- Traditional Machine Learning Algorithms
- Deep learning
- Probabilistic Graphical Models
    - Introduction
    - I-Map, Perfect Map
    - Reasoning Patterns (Causal Reasoning, Evidential Reasoning, Intercausal Reasoning)



Recap: Local Independencies in a BN

- A BN G is a directed acyclic graph whose nodes represent random variables $X_i, \ldots, X_n$.
- Let $Pa(X_i)$ denote parents of X_i in G
- Let $\textit{Non-Desc}(X_i)$ denote variables in G that are not descendants of X_i
- Then G encodes the following set of *conditional independence* assumptions denoted *I*l(G)
  - For each X_i: $(X_i \perp \textit{Non-Desc}(X_i) \mid Pa(X_i))$
- Also known as *Local Markov Independencies*

Recap: Local Independencies



- Graph G with CPDs is equivalent to a set of independence assertions

$$P(D,I,G,S,L) = P(D)P(I)P(G \mid D,I)P(S \mid I)P(L \mid G)$$

- Local Conditional Independence Assertions (starting from leaf nodes):

$I(G) = \{(L \perp I,D,S \mid G),$	L is conditionally independent of all other nodes given parent G
$(S \perp D,G,L \mid I),$	S is conditionally independent of all other nodes given parent I
$(G \perp S \mid D,I),$	Even given parents, G is NOT independent of descendant L
$(\mathrm{I} \perp D \mid \phi),$	Nodes with no parents are marginally independent
$(D \perp I,S \mid \phi)\}$	D is independent of non-descendants I and S

- Parents of a variable shield it from probabilistic influence
  - Once value of parents known, no influence of ancestors
- Information about descendants can change beliefs about a node

Can we have the following conditional independence?

$$D \perp S \mid G$$

$$D \perp S \mid I$$

$$D \perp S \mid G, I$$

Recap:

low grade drastically decreases the probability of high intelligence

low grade justifies the difficulty

$P(i^1)=0.3$

$P(i^1 \,|\, g^3)=0.079$

$P(d^1 \,|\, g^3)=0.629$

$P(i^1 \,|\, l^0)=0.14$

$P(i^1 \,|\, l^0, g^3)=0.079$

A weak letter drastically decreases the probability of high intelligence

After knowing low grade, a weak letter won't make the probability of high intelligence lower.

$$I \perp L \,|\, G$$

Independencies in Graphs

- A graph structure G encodes a set of conditional independence assumptions *I*(G)
- Are there other independencies that we can read-off?
  - i.e., are there other independencies that hold for every distribution that factorizes over G?
- D-separation holds the key

Topics

- Why D-separation?
- What is D-separation?
- Algorithm for D-separation (extended materials)

Why D-separation?

Dependencies and Independencies

- Crucial for understanding network behaviour
- Independence properties are important for answering queries
  - Exploited to reduce computation of inference
  - A distribution *P* that factorizes over *G* satisfies *I(G)*

What is D-separation?

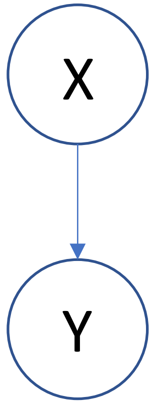
D-separation

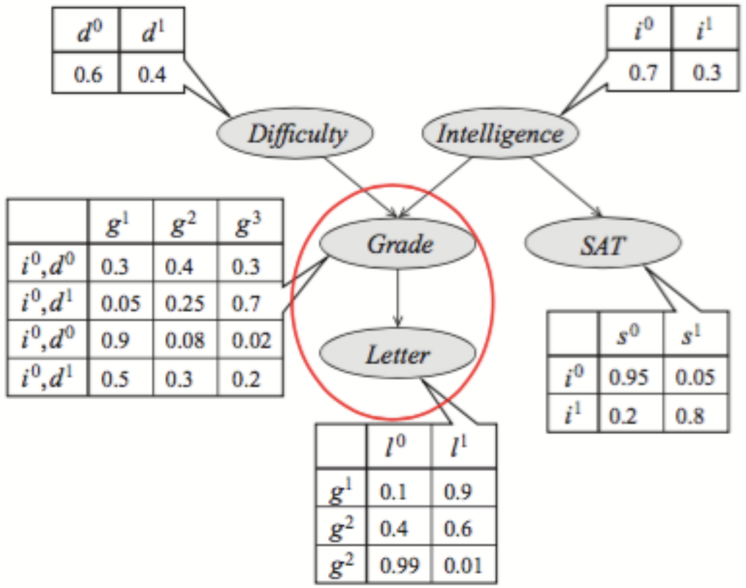
- Study independence properties for subgraphs (connected triples)
- Analyze complex cases in terms of triples along paths between variables
- ***D-separation:*** a condition / algorithm for answering such queries

- Definition: A procedure $\text{d-sep}_G(X \perp Y | Z)$ that given a DAG *G*, and sets *X*, *Y*, and *Z* returns either *yes* or *no, where* $\text{d-sep}_G(X \perp Y | Z) = \text{Yes}$ iff $(X \perp Y | Z)$ follows from *I(G)*

Direct Connection between *X* and *Y*

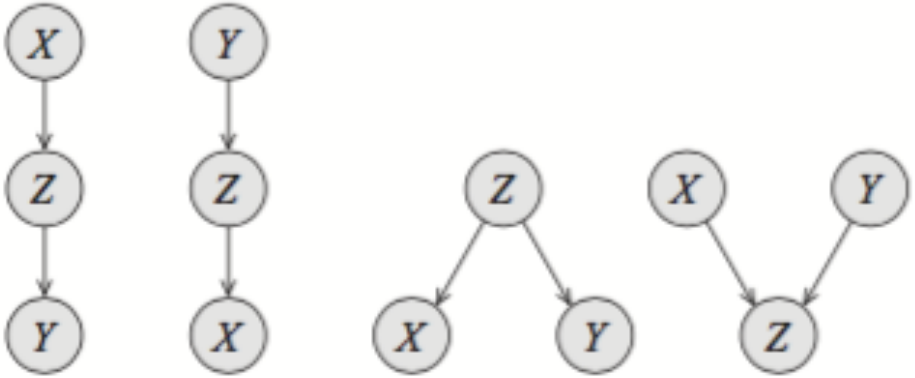
- *X* and *Y* are correlated regardless of any evidence about any other variables
  - E.g., Feature *Y* and character *X* are correlated
  - Grade *G* and Letter *L* are correlated
- If *X* and *Y* are directly connected we can get examples where they influence each other regardless of *Z*





Indirect Connection between X and Y

- Four cases where X and Y are connected via Z
- (a). Indirect causal effect
- (b). Indirect evidential effect
- (c). Common cause
- (d). Common effect

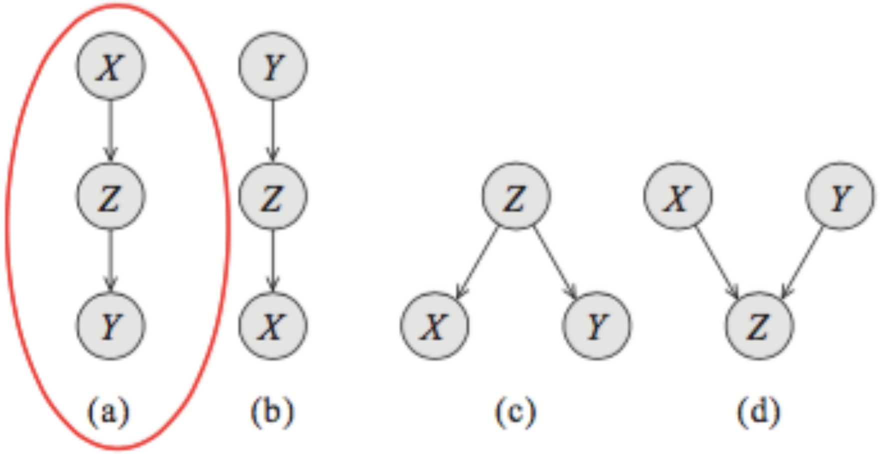


(a) (b) (c) (d)

- We will see that first three cases are similar while fourth case (V-structure) is different

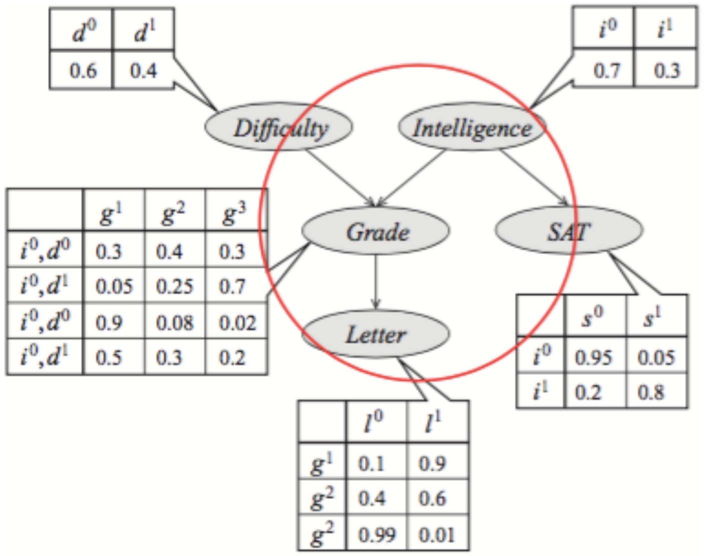
1. Indirect Causal Effect: *X->Z->Y*

- Cause *X* cannot influence effect *Y* if *Z* observed
  - Observed *Z* blocks influence
- If *Grade* observed then *I* does not influence *L*
  - *Intelligence* influences *Letter* if *Grade* is unobserved



$$Z = Grade$$

$$I \perp L \mid G$$



d^0	d^1
0.6	0.4

i^0	i^1
0.7	0.3

	g^1	g^2	g^3
i^0,d^0	0.3	0.4	0.3
i^0,d^1	0.05	0.25	0.7
i^0,d^0	0.9	0.08	0.02
i^0,d^1	0.5	0.3	0.2

	s^0	s^1
i^0	0.95	0.05
i^1	0.2	0.8

	l^0	l^1
g^1	0.1	0.9
g^2	0.4	0.6
g^2	0.99	0.01

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low grade justifies the difficulty

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$$I \perp L \mid G$$

Causal Chains

- This configuration is a "causal chain"

X: Low pressure Y: Rain Z: Traffic

$$P(x,y,z) = P(x)P(y|x)P(z|y)$$

Guaranteed X independent of Z given Y?

$$P(z|x,y) = \frac{P(x,y,z)}{P(x,y)}$$

$$= \frac{\cancel{P(x)P(y|x)}P(z|y)}{\cancel{P(x)P(y|x)}}$$

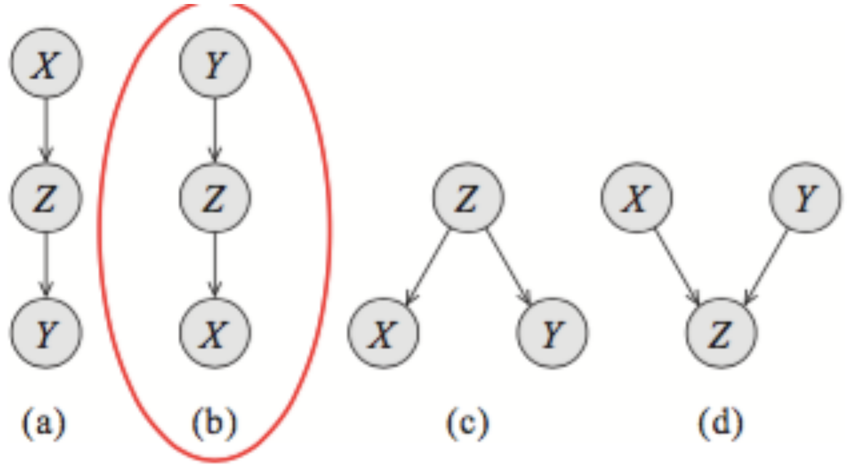
$$= P(z|y)$$

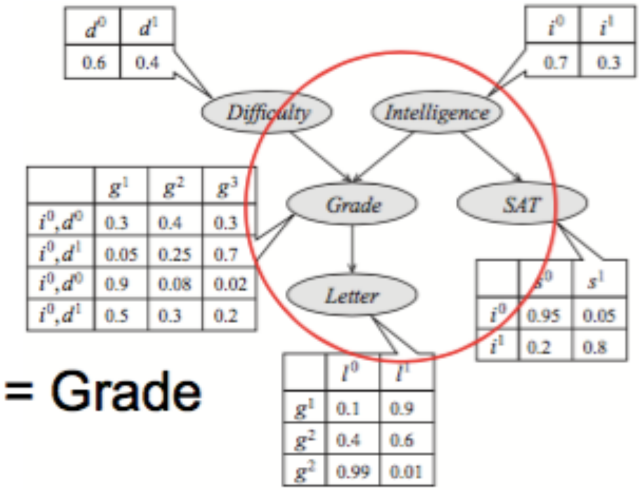
Yes!

Evidence along the chain "blocks" the influence (makes "inactive")

2. Indirect Evidential Effect: *Y->Z->X*

- Evidence *X* can influence *Y* via *Z* only if *Z* is unobserved
  - Observed *Z* blocks influence
- If Grade unobserved, Letter influences assessment of Intelligence
- Dependency is a symmetric notion
  - $X \perp Y$ does not hold then $Y \perp X$ does not hold either

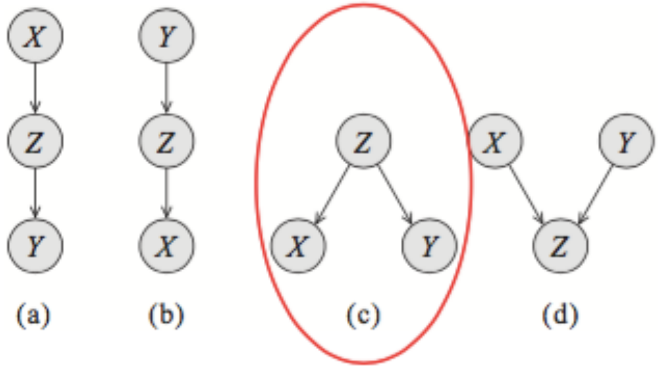




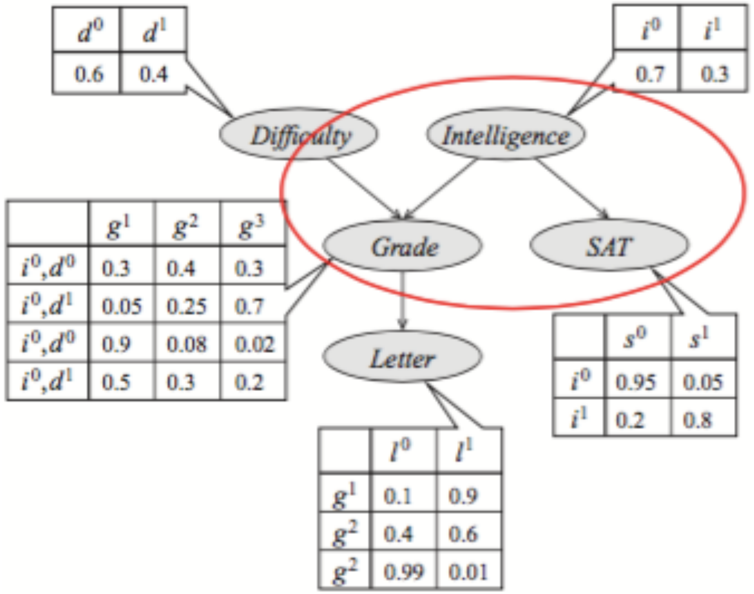
Z = Grade

3. Common Cause: *X<-Z->Y*

- *X* can influence *Y* if and only if *Z* is not observed
  - Observed *Z* blocks
- Grade is correlated with SAT score
- But if Intelligence is observed then SAT provides no additional information

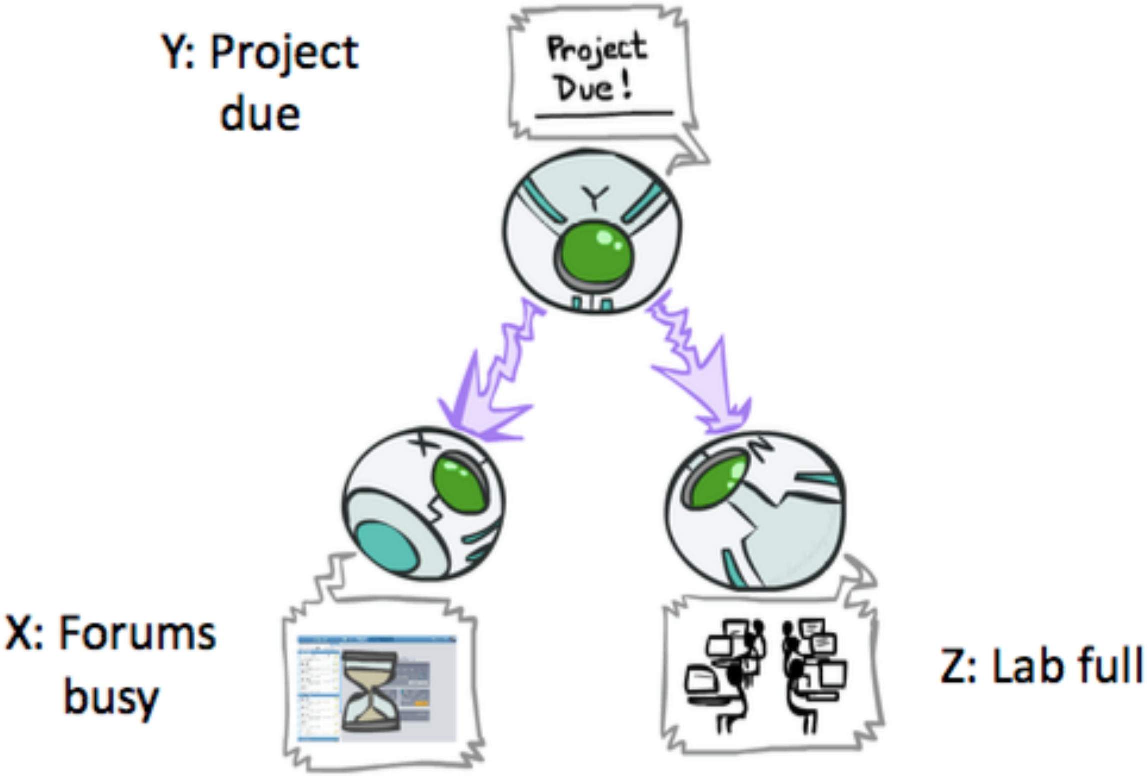


$$(S \perp G \mid I)$$



Common Cause

- This configuration is a "common cause"



$$P(x, y, z) = P(y)P(x|y)P(z|y)$$

- Guaranteed X and Z independent given Y?

$$P(z|x, y) = \frac{P(x, y, z)}{P(x, y)}$$

$$= \frac{\cancel{P(y)}\cancel{P(x|y)}P(z|y)}{\cancel{P(y)}\cancel{P(x|y)}}$$

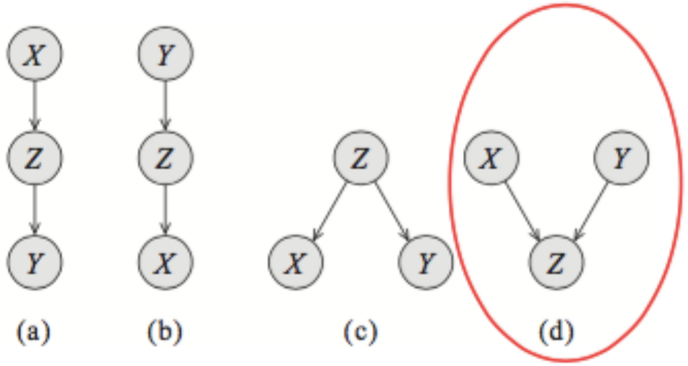
$$= P(z|y)$$

Yes!

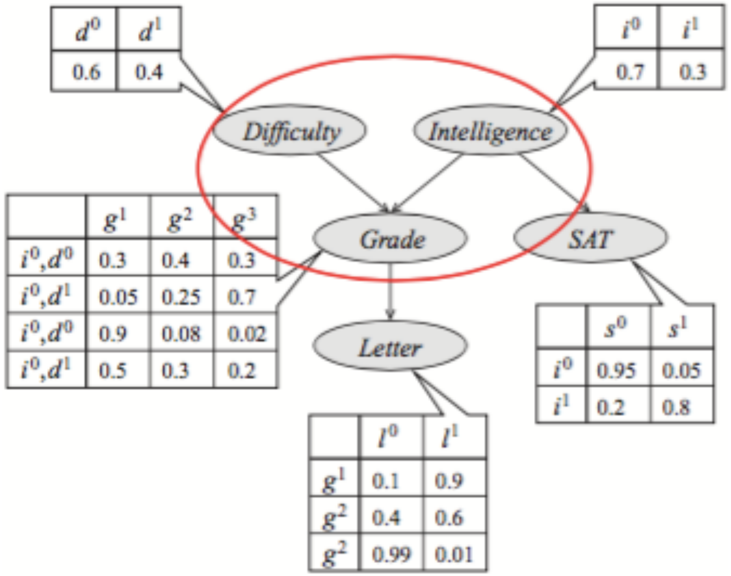
Observing the cause blocks influence between effects. (makes inactive)

4. Common Effect (V-structure) *X->Z<-Y*

- Influence cannot flow on trail *X->Z<-Y* if *Z* is not observed
  - Observed *Z* enables
  - Opposite to previous 3 cases (Observed *Z* blocks)
- When *G* not observed *I* and *D* are independent
- When *G* is observed, *I* and *D* are correlated

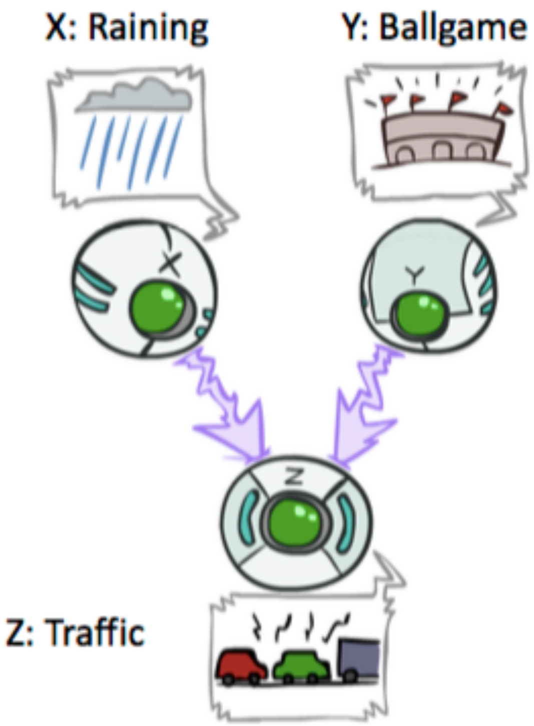


$$I \perp D | {\sim} G$$



Common Effect

- Last configuration: two causes of one effect (v-structures)

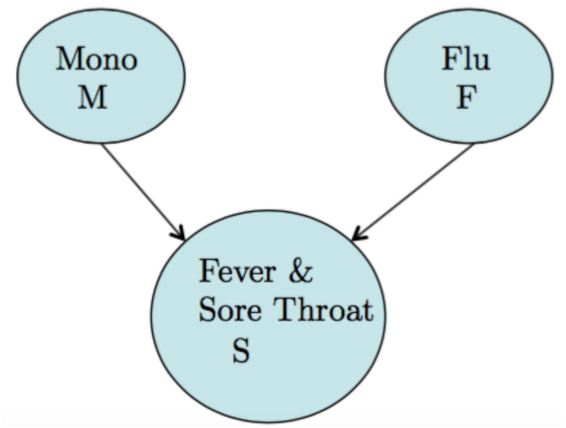


- Are X and Y independent?
  - *Yes*: the ballgame and the rain cause traffic, but they are not correlated
  - Still need to prove they must be (try it!)
- Are X and Y independent given Z?
  - *No*: seeing traffic puts the rain and the ballgame in competition as explanation.
- This is ***backwards*** from the other cases
  - Observing an effect activates influence between possible causes. (makes active!)

Recall: Common in Human Reasoning

- Binary Variables
- Fever & Sore Throat can be caused by mono and flu
- When flu is diagnosed probability of mono is reduced (although mono could still be present)
- It provides an alternative explanation of symptoms

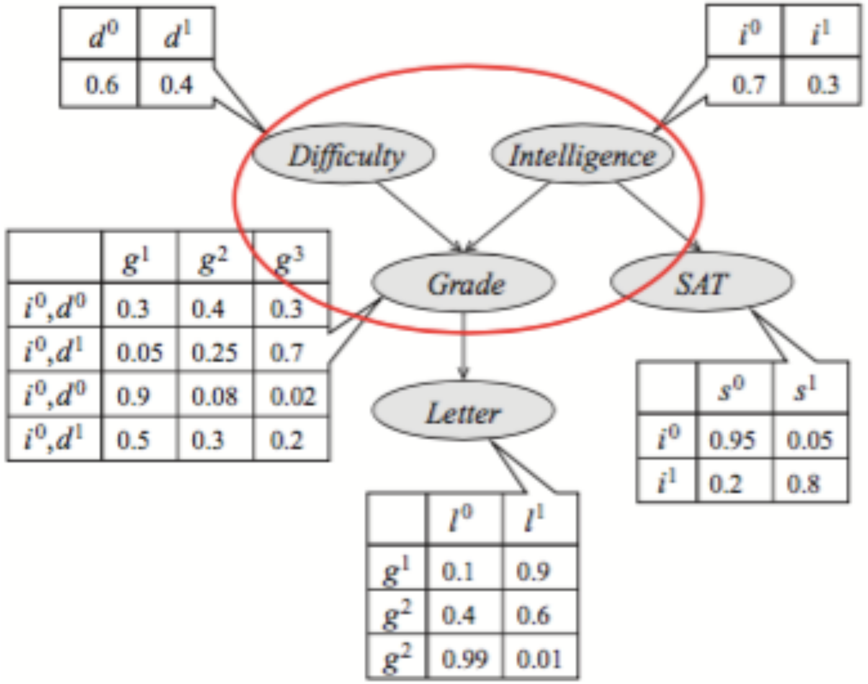
$$P(m^1 | s^1) > P(m^1 | s^1, f^1)$$



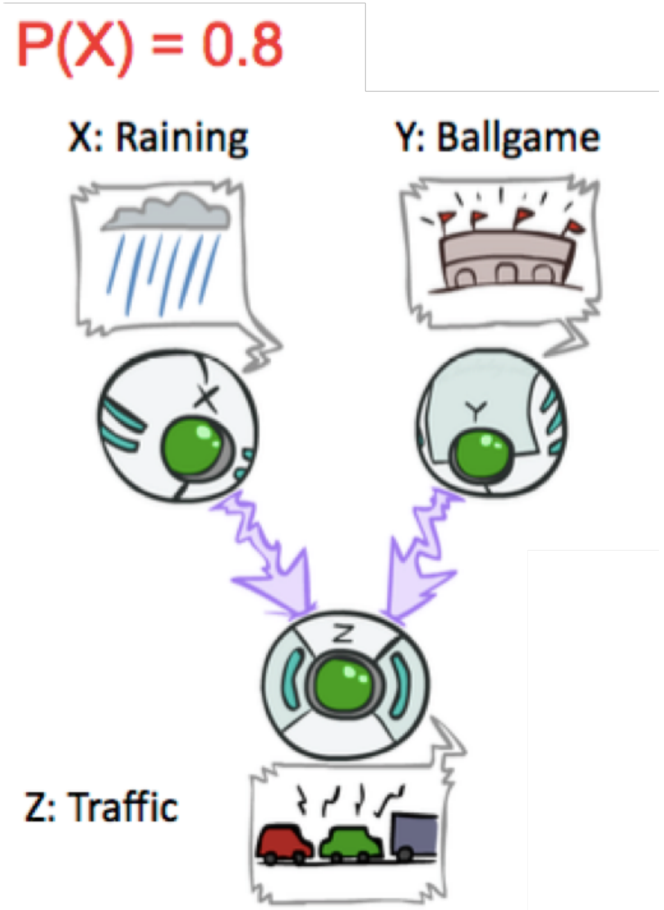
4. Common Effect (V-structure) *X->Z<-Y*

- *Grade* is not observed
- Observe weak letter *L*
  - Which indicates low *Grade*
  - Suffices to correlate *D* and *I*

Not child, but
descendants

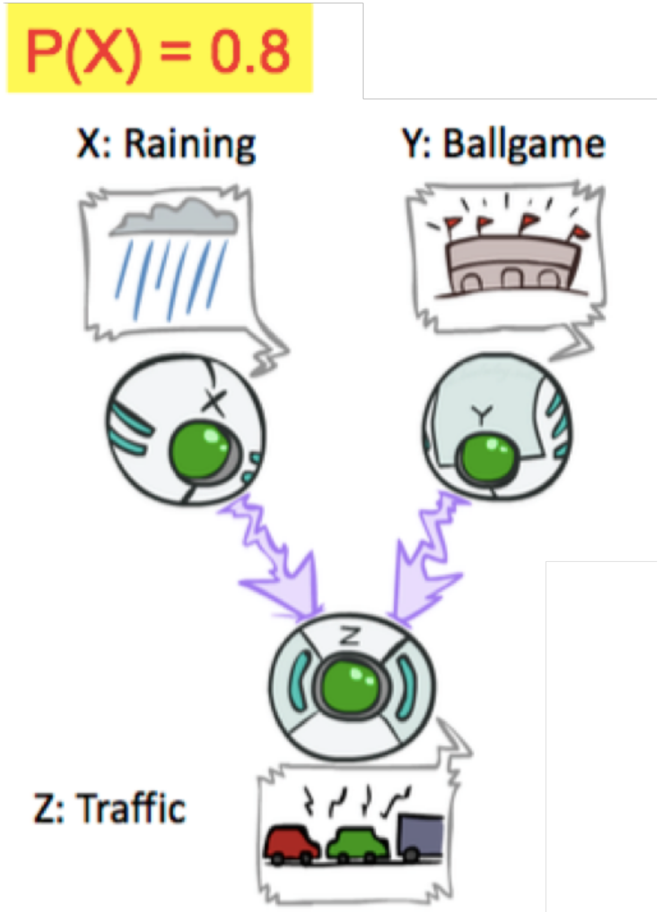


Example: Common Effect



X	Y	Z	P
T	T	T	0.076
T	T	F	0.004
T	F	T	0.576
T	F	F	0.144
F	T	T	0.162
F	T	F	0.002
F	F	T	0.090
F	F	F	0.009

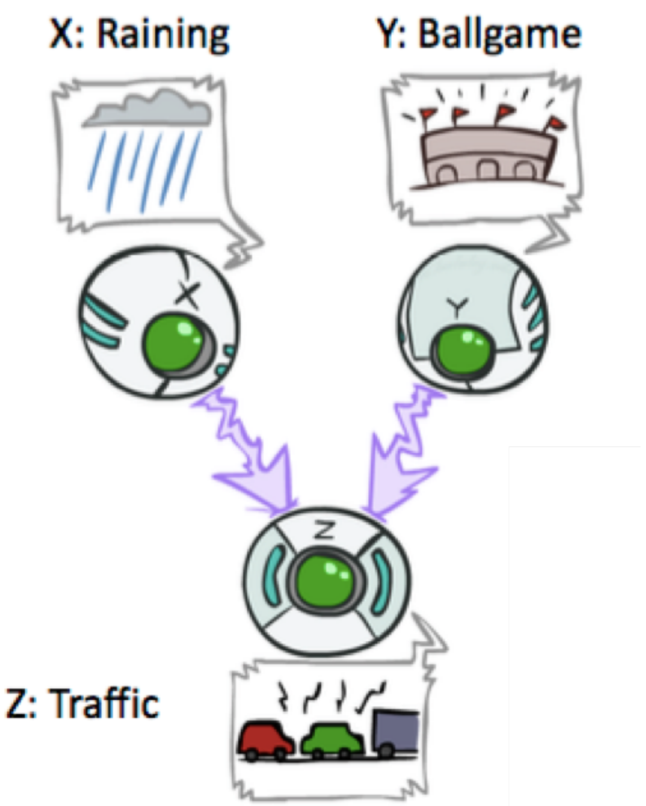
Example: Common Effect



X	Y	Z	P
T	T	T	0.076
T	T	F	0.004
T	F	T	0.576
T	F	F	0.144
F	T	T	0.162
F	T	F	0.002
F	F	T	0.090
F	F	F	0.009

Example: Common Effect

$P(X) = 0.8$



X	Y	Z	P
T	T	T	0.076
T	T	F	0.004
T	F	T	0.576
T	F	F	0.144
F	T	T	0.018
F	T	F	0.002
F	F	T	0.090
F	F	F	0.009

$$P(X|Y) = \frac{0.076+0.004}{0.076+0.004+0.018+0.002}$$

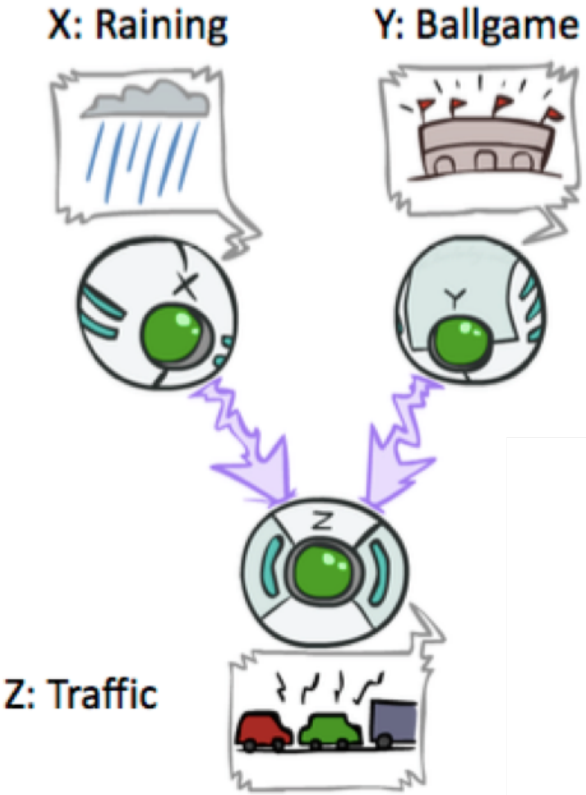
$$= 0.08 / 0.1$$

$$= 0.8$$

X and Y are independent!

But Suppose Also Know Z=T

P(X) = 0.8



$$P(X|Z) = \frac{.076+.576}{.076+.576+.018+.090}$$
$$= 0.652/0.76$$
$$= 0.858$$

$$P(X|Y,Z) = \frac{0.076}{0.076+0.018}$$
$$= 0.8085$$

X	Y	Z	P
T	T	T	0.076
T	T	F	0.004
T	F	T	0.576
T	F	F	0.144
F	T	T	0.018
F	T	F	0.002
F	F	T	0.090
F	F	F	0.009

X and Y are not independent given Z!

Summary of Indirect Connection

- Causal trail: *X->Z->Y*: active iff *Z* not observed
- Evidential Trail: *X<-Z<-Y*: active iff *Z* is not observed
- Common Cause: *X<-Z->Y*: active iff *Z* is not observed
- Common Effect: *X->Z<-Y*: active iff either *Z* or one of its descendants is observed

What is the general case?

The General Case

- General question: in a given BN, are two variables independent (given evidence)?
- Solution: analyze the graph
- Any complex example can be broken into repetitions of the three canonical cases

D-Separation

- Query: $X_i \perp X_j | \{X_{k_1}, ..., X_{k_n}\}$?
- Check all (undirected) paths between X_i and X_j
  - If one or more paths is active, then independence not guaranteed

$$X_i \not\perp X_j | \{X_{k_1}, ..., X_{k_n}\}$$

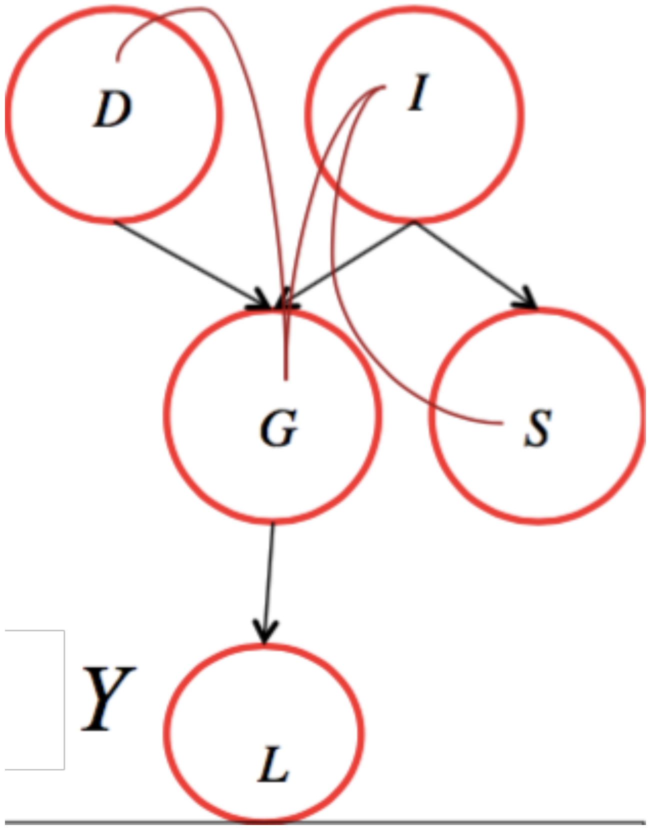
  - Otherwise (i.e. if ***all paths*** are inactive),
    then "D-separated" = independence ***is*** guaranteed

$$X_i \perp X_j | \{X_{k_1}, ..., X_{k_n}\}$$

Active Trail

- When influence can flow from *X* to *Y* via *Z* then trail *X*—*Z*—*Y* is active

- Example: Consider Trail *D*->*G*<-*I*->*S*
- When observed
  - $Z = \{\emptyset\}$ *:* trail is *inactive* because v-structure *D*->*G*<-*I* is inactive
  - $Z = \{L\}$ *: active* (*D*->*G*<-*I* active) since L is descendant of G
  - $Z = \{L, I\}$ *: inactive* because observing I blocks *G*<-*I*->*S*



D-separation definition

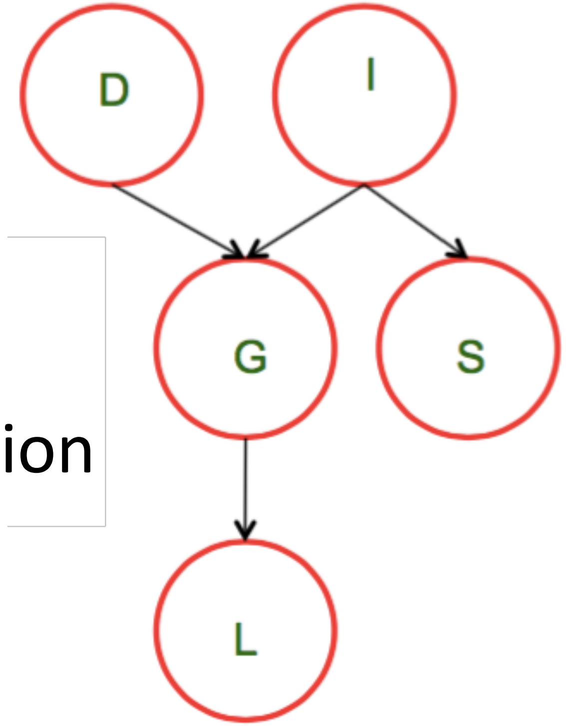
- Let ***X*,*Y*** and ***Z*** be three sets of nodes in G.
- ***X*** and ***Y*** are d-separated given ***Z*** denoted $\text{d-sep}_G(X \perp Y | Z)$ if there is no active trail between any node $X \in \boldsymbol{X}$ and $Y \in \boldsymbol{Y}$ given ***Z***
- That is, nodes in ***X*** cannot influence nodes in ***Y***
- Provides notion of separation between nodes in a directed graph ("directed" separation)

Independencies from D-separation

- Definition

$$I(G) = \{(X \perp Y | Z) : \text{d-sep}_G(X \perp Y | Z)\}$$

  - Also called *Global Markov independencies*
  - Note: Derived purely from graph (using trails)
- Example: *Global* independence using D-separation
  - $(L \perp I, D, S | G) \in I(G)$



- Compare with *local* independencies
  - For each X_i: ($X_i \perp$ Non-Desc(X_i) | $Pa(X_i)$)

$$\{(L \perp I, D, S | G), (S \perp D, G, L | I), (G, S | D, I), (D \perp I, S | \emptyset)\} \subseteq I(G)$$

Algorithm for D-Separation

Algorithm for D-Separation

- Enumerate all trails between *X* & *Y* is inefficient
  - No. of trails is *exponential* with graph size
- Linear time algorithm has two phases
  - Algorithm *Reachable*(G,*X*,***Z***) returns nodes reachable from *X* given ***Z***
  - Phase 1 (simple)
    - Traverse bottom-up from leaves marking all nodes in ***Z*** or descendants in ***Z***; to enable v-structures
  - Phase 2 (subtle)
    - Traverse top-down from *X* to *Y,* stopping when blocked by a node

Structure Implications

- Given a Bayes net structure, can run d-separation algorithm to build a complete list of conditional independences that are necessarily true of the form

$$X_i \perp X_j | \{X_{k_1}, ..., X_{k_n}\}$$

- This list determines the set of probability distributions that can be represented

Pseudocode

finding nodes reachable from *X* given ***Z*** via active trails

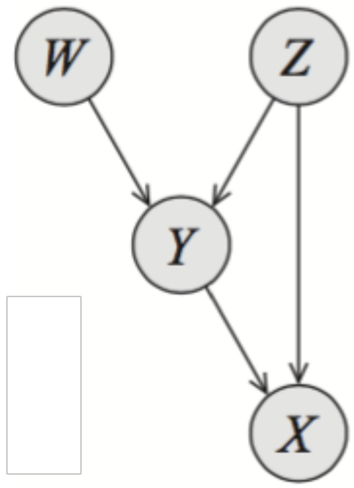
```
Procedure Reachable (
    G,   // Bayesian network graph
    X,   // Source variable
    Z    // Observations
)
    // Phase I: Insert all ancestors of Z into V
    L ← Z    // Nodes to be visited
    A ← ∅    // Ancestors of Z
    while L ≠ ∅
      Select some Y from L
      L ← L − {Y}
      if Y ∉ A then
        L ← L ∪ Pa_Y    // Y's parents need to be visited
      A ← A ∪ {Y}    // Y is ancestor of evidence

    // Phase II: traverse active trails starting from X
    L ← {(X, ↑)}    // (Node,direction) to be visited
    V ← ∅    // (Node,direction) marked as visited
    R ← ∅    // Nodes reachable via active trail
    while L ≠ ∅
      Select some (Y, d) from L
      L ← L − {(Y, d)}
      if (Y, d) ∉ V then
        if Y ∉ Z then
          R ← R ∪ {Y}    // Y is reachable
        V ← V ∪ {(Y, d)}    // Mark (Y, d) as visited
        if d = ↑ and Y ∉ Z then    // Trail up through Y active if Y not in Z
          for each Z ∈ Pa_Y
            L ← L ∪ {(Z, ↑)}    // Y's parents to be visited from bottom
          for each Z ∈ Ch_Y
            L ← L ∪ {(Z, ↓)}    // Y's children to be visited from top
        else if d = ↓ then    // Trails down through Y
          if Y ∉ Z then
            // Downward trails to Y's children are active
            for each Z ∈ Ch_Y
              L ← L ∪ {(Z, ↓)}    // Y's children to be visited from top
          if Y ∈ A then    // v-structure trails are active
            for each Z ∈ Pa_Y
              L ← L ∪ {(Z, ↑)}    // Y's parents to be visited from bottom
  return R
```

Example for D-separation algorithm

- Task: Find all nodes reachable from *X*
- Assume that *Y* is observed, i.e., $Y \in \boldsymbol{Z}$
- Assume algorithm first encounters *Y* via edge *Y* -> *X*
- Any extension of this trail is blocked by *Y*
- Trail *X*<-*Z*->*Y*<-*W* is not blocked by *Y*
- Thus when we encounter *Y* for the second time via the edge *Z*->*Y* we should not ignore it
- Therefore after the first visit to *Y* we can mark it as visited

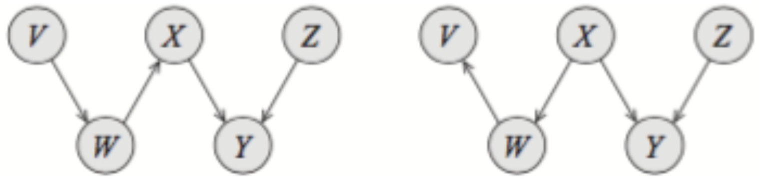
For trails coming from children of *Y*
Not for purpose of trails coming from parents of *Y*



I-Equivalence

I-Equivalence

- Conditional independence assertion statements can be the same with different structures
- Two graphs G_1 and G_2 are *I- equivalent* if $I(G_1)=I(G_2)$
- Skeleton of a BN graph G is an undirected graph with an edge for every edge in G
- If two BN graphs have the same set of skeletons and v-structures then they are *I-equivalent*



Same skeleton
Same v-structure $X \rightarrow Y \leftarrow Z$

Bayes Nets Representation Summary

- Bayes nets compactly encode joint distributions
- Guaranteed independencies of distributions can be deduced from BN graph structure
- D-separation gives precise conditional independence guarantees from graph alone
- A Bayes’ net’s joint distribution may have further (conditional) independence that is not detectable until you inspect its specific distribution