From LTL to rLTL Monitoring: Improved Monitorability through Robust Semantics*

Corto Mascle[1], Daniel Neider[2], Maximilian Schwenger[3], Paulo Tabuada[4], Alexander Weinert[5], and Martin Zimmermann[6]

[1] ENS Paris-Saclay, Cachan, France
[2] Max Planck Institute for Software Systems, Kaiserslautern, Germany
neider@mpi-sws.org
[3] Reactive Systems Group, Saarland University, Saarbrücken, Germany
[4] Department of Electrical and Computer Engineering, UCLA, Los Angeles, USA
[5] German Aerospace Center (DLR), Cologne, Germany
[6] University of Liverpool, Liverpool, United Kingdom

Abstract. Runtime monitoring is commonly used to detect the violation of desired properties in safety critical systems by observing run prefixes of the system. Bauer et al. introduced an influential framework for monitoring Linear Temporal Logic (LTL) properties, which is based on a three-valued semantics: the formula is already satisfied by the given prefix, it is already violated, or it is still undetermined, i.e., it can be satisfied and violated. However, a wide range of formulas are not monitorable under this approach, meaning that every prefix is undetermined. In particular, Bauer et al. report that 44% of the formulas they consider in their experiments fall into this category.
Recently, robust semantics for LTL were introduced to capture degrees of violation of universal properties. Here, we define robust semantics for run prefixes and show its potential in monitoring: every formula considered by Bauer et al. is monitorable under our approach. Furthermore, we show that properties expressed with the robust semantics can be monitored by deterministic automata.

Keywords: Runtime Monitoring · Linear Temporal Logic · Robustness

1 Introduction

Runtime monitoring is nowadays routinely used to assess the satisfaction of properties of systems during their execution. It is especially useful for systems that cannot be verified prior to deployment and, for this reason, can contain hidden bugs. While it is useful to catch and document these bugs during an execution of a system, we find that the current approach to runtime verification based on Linear Temporal Logic (LTL) [14] is not sufficiently informative, especially in what regards a system's robustness. Imagine that we are monitoring a property φ and that this property is violated during an execution. In addition to be alerted to the presence of a bug, there are several other questions we would like to have answered such as: although φ was falsified, was there a *weaker* version of φ that was still satisfied or did the system fail catastrophically? Similarly, if we consider a property of the form $\varphi \rightarrow \psi$, where φ is an environment assumption and ψ is a system guarantee, and the environment violates φ *slightly* along an execution can we still guarantee that ψ is only *slightly* violated?

Answering these questions requires a logical formalism for specifying properties that provides meaning to terms such as *weaker* and *slightly*. Formalizing these notions within temporal logic, so as to be able to reason about the robustness of a system, was the main impetus behind the definition of *robust Linear-time Temporal Logic* (rLTL) [50]. While reasoning in LTL yields a binary result, rLTL adopts a five-valued semantics representing different *shades of violation*. Consider, for example, the specification $\Box a \rightarrow \Box b$ requiring that b

* The work of Paulo Tabuada was partially supported by the NSF project 1645824. The work of Alexander Weinert was supported by the Saarbrücken Graduate School of Computer Science. The work of Maximilian Schwenger was supported by the European Research Council (ERC) Grant OSARES (No. 683300).

is always satisfied provided a is always satisfied. In LTL, if the premise a is violated in a single position of the trace, then the specification is satisfied vacuously, eliminating all requirements on the system regarding $\Box b$. In this case, rLTL detects a mild violation of the premise and thus allows for a mild violation of the conclusion.

The very same reasons that make runtime verification for LTL so useful also motivate the need for developing the theoretical framework proposed in this paper for rLTL runtime verification. To this end, we tackle the problem of evaluating a property over infinite traces based on a finite prefix similarly to Bauer et al. [14]. If the available information is insufficient to declare a specification violated or satisfied, the monitor reports a ?. This concept is applied to each degree of violation of the rLTL semantics. Thus, the rLTL monitor's verdict consists of four three-valued bits. Each bit represents a degree of violation/satisfaction of the specification in increasing order of severity.

As an example, consider an autonomous drone that may or may not be in a stable state[7]. The specification requires that it remains stable throughout the entire mission. However, if the take-off is shaky due to bad weather, the drone is unstable for the first couple of minutes of the mission. An LTL monitor thus jumps to the conclusion that the specification is violated whereas an rLTL monitor only reports a *partial* violation. As soon as the drone stabilizes, the rLTL monitor refines its verdict by also reporting a partial satisfaction.

The main research contributions of this paper are thus a finite trace semantics for rLTL coupled with an algorithm to synthesize monitors for rLTL specifications. The construction is doubly-exponential in the size of the formula, so rLTL monitoring is no more costly than LTL monitoring.

We validate our approach by analyzing empirically whether an rLTL monitor provides more information than an LTL monitor. As performance metric, we use Bauer et al.'s [13] definition of *monitorability*. An LTL formula is monitorable if for any infinite trace there is a finite prefix by which the monitor can decide whether or not the formula is violated or satisfied. We adapt this definition to rLTL and replicate Bauer et al.'s experiments. We have found that 44% of their formulas are not LTL-monitorable whereas all of them are rLTL-monitorable. This indicates that rLTL monitoring is an improvement over LTL monitoring in terms of monitorability.

Proofs omitted due to space constraints can be found in the appendix.

Related Work In runtime verification [21,33,38,44] the specification is often given in LTL [42]. While properties arguing about the past or current state of a system are easily monitorable [32], LTL can also express assumptions on the future that cannot be validated using only a finite prefix of a word. Thus, adaptations of LTL have been proposed which include different notions of a next step on finite words [23,41], lifting LTL to a three- or four-valued domain [13,14], or applying predictive measures to rule out impossible extensions of words [51].

Moreover, tools for monitoring first-order temporal logics using BDDs [31], Regular Expressions [11], SMT solvers [19] or rule-based systems [9,10] have been introduced. These, however, only yield binary results.

Non-binary monitoring has been addressed by adding quantitative measures such as counting events [8,43]. Moreover, monitoring tools collecting statistics [2,4,28] become increasingly popular: Snort [48] is a commercial tool for rule-based network monitoring and computing efficient statistics, Beep Beep 3 [30] is a tool based on a query language allowing for powerful aggregation functions and statistical measures. On the downside, it imposes the overhead of running a heavy-weight application on the monitored system. In contrast, we generate monitor automata out of an rLTL formula. Such an automaton can easily and automatically be implemented on almost any system with statically determined memory requirements and negligible performance overhead. Similarly, the Copilot [46] framework based on synchronous languages [16,17] transforms a specification in a declarative data-flow language into a C implementation of a monitor with constant space and time requirements. Lola [3,17] allows for more involved computations, also incorporating parametrization [26] and real-time capabilities [25] while retaining constant space and time requirements.

Rather than enriching temporal logics with quantitative measures [35], we consider a robust version of LTL: rLTL [6,5,50]. Robust semantics yields information about to which degree a trace violates a property. We adapt the semantics to work with finite traces by allowing for intermediate verdicts. Here, a certain

[7] By this we mean, e.g., that the error in tracking a desired trajectory is below a certain threshold.

degree of violation can be classified as "indefinite" and refined when more information becomes available to the monitor. In a similar fashion, for Signal Temporal Logic [40,39], Donze et al. [20] introduced a notion of robustness. Here, the output is real-valued which comes at a prize of significantly higher overhead at runtime. The quality of the monitor's output can be improved by distinguishing between input and output signals [27] or taking the signal processing pipeline into account [1].

2 Robust Linear Temporal Logic

Throughout this work, we assume basic familiarity with classical LTL and refer the reader to a textbook for more details on the logic (see, e.g., [7]). Moreover, let us fix some finite set P of atomic propositions throughout the paper and define $\Sigma = 2^P$. We denote the set of finite and infinite words over Σ by Σ^* and Σ^ω, respectively. The empty word is denoted by ε and $\sqsubseteq$ and $\sqsubset$ denote the non-strict and the strict prefix relation, respectively. Moreover, we denote the set of Booleans by $\mathbb{B} = \{0, 1\}$.

The logics LTL and rLTL share the same syntax save for a dot superimposed on temporal operators. More precisely, the syntax of rLTL is given by the grammar

$$\varphi \coloneqq p \mid \neg\varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \varphi \rightarrow \varphi \mid \odot\varphi \mid \varphi \,\dot{\mathbf{U}}\, \varphi \mid \varphi \,\dot{\mathbf{R}}\, \varphi \mid \dot{\Diamond}\varphi \mid \boxdot\varphi,$$

where p ranges over atomic propositions in P and the temporal operators $\odot$, $\dot{\mathbf{U}}$, $\dot{\mathbf{R}}$, $\dot{\Diamond}$ and $\boxdot$ correspond to "next", "until", "release", "eventually", and "always", respectively. [8] The size $|\varphi|$ of a formula φ is the number of its distinct subformulas. Furthermore, we denote the set of all LTL and rLTL formulas over P by Φ_{LTL} and Φ_{rLTL}, respectively.

The development of rLTL was motivated by the observation that the difference between "minor" and "major" violations of a formula cannot be adequately described in a two-valued semantics. If an LTL formula φ, for example, demands that the property p holds at all positions of a word $\sigma \in \Sigma^\omega$, then σ violates φ even if p does not hold at only a single position, a very minor violation. The semantics of LTL, however, does not differentiate between the σ above and a σ' in which the property p never holds, a major violation of the property φ.

In order to alleviate this shortcoming, Tabuada and Neider introduced Robust Linear-time Temporal Logic (rLTL) [50], whose semantics allows for distinguishing various "degrees" to which a word violates a formula. More precisely, the semantics of rLTL are defined over the set $\mathbb{B}_4 = \{0000, 0001, 0011, 0111, 1111\}$ of five *truth values*, each of which is a monotonically increasing sequence of four bits. The truth values are interpreted as follows: 0000 corresponds to *false*, 1111 to *true*, and the remaining truth values to different "shades" of false, reflecting degrees of violation of a formula. To be able to refer to individual bits of an rLTL truth value $\beta \in \mathbb{B}_4$, we use $\beta[i]$ with $i \in \{1, \dots, 4\}$ as to denote the i-th bit of β.

For the sake of a simpler presentation, we denote the semantics of both LTL and rLTL not in terms of satisfaction relations but by means of *valuation functions*. For LTL, the valuation function $V \colon \Sigma^\omega \times \Phi_{LTL} \rightarrow \mathbb{B}$ assigns to each infinite word $\sigma \in \Sigma^\omega$ and each LTL formula $\varphi \in \Phi_{LTL}$ the value 1 if σ satisfies φ and the value 0 if σ does not satisfy φ, and is defined as usual (see, e.g., [7]). The semantics of rLTL, on the other hand, is more complex and formalized next.

Definition 1 ([5,6,50]). *Let the function* $\mathrm{ltl} \colon \{1, \dots, 4\} \times \Phi_{rLTL} \rightarrow \Phi_{LTL}$ *be inductively defined as in Table 1. The rLTL semantics is then given as the valuation function* $V_r \colon \Sigma^\omega \times \Phi_{rLTL} \rightarrow \mathbb{B}_4$*, where for every* $\sigma \in \Sigma^\omega$*, every rLTL formula* φ*, and every* $i \in \{1, \dots, 4\}$*, the* i*-th bit of* $V_r(\sigma, \varphi)$ *is defined as* $V_r(\sigma, \varphi)[i] = V\big(\sigma, \mathrm{ltl}(i, \varphi)\big)$ *(i.e., via the semantics of the LTL formulas* $\mathrm{ltl}(i, \varphi)$*).*

Intuitively, the four bits denote distinct levels of the violation of a property. For instance, for an rLTL formula $\boxdot\varphi$ and a word $\sigma \in \Sigma^\omega$, the rLTL semantics ensures that (a) $V_r(\sigma, \boxdot\varphi)[1] = 1$ if φ holds at all positions of σ, (b) $V_r(\sigma, \boxdot\varphi)[2] = 1$ if φ holds at almost all positions of σ, (c) $V_r(\sigma, \boxdot\varphi)[3] = 1$ if φ holds

[8] Note that we include the operators $\wedge$, $\rightarrow$, and $\dot{\mathbf{R}}$ explicitly in the syntax as they cannot be derived from other operators due to the many-valued nature of rLTL. Following the original work on rLTL [50], we also include the operators $\dot{\Diamond}$ and $\boxdot$ explicitly (which can be derived from $\dot{\mathbf{U}}$ and $\dot{\mathbf{R}}$, respectively).

Table 1. The function ltl: $\{1,\ldots,4\} \times \Phi_{rLTL} \to \Phi_{LTL}$.

Operator	*Symbol*	*Semantics (where $p \in P$ and $\varphi, \psi \in \Phi_{rLTL}$)*
Atomic proposition	$p \in P$	$1 \le i \le 4$: $\mathrm{ltl}(i,p) = p$
Negation	$\neg$	$1 \le i \le 4$: $\mathrm{ltl}(i,\neg\varphi) \coloneqq \neg\mathrm{ltl}(1,\varphi)$
Disjunction	$\vee$	$1 \le i \le 4$: $\mathrm{ltl}(i,\varphi \vee \psi) \coloneqq \mathrm{ltl}(i,\varphi) \vee \mathrm{ltl}(i,\psi)$
Conjunction	$\wedge$	$1 \le i \le 4$: $\mathrm{ltl}(i,\varphi \wedge \psi) \coloneqq \mathrm{ltl}(i,\varphi) \wedge \mathrm{ltl}(i,\psi)$
Implication	$\to$	$1 \le i \le 3$: $\mathrm{ltl}(i,\varphi \to \psi) \coloneqq (\mathrm{ltl}(i,\varphi) \to \mathrm{ltl}(i,\psi)) \wedge \mathrm{ltl}(i+1,\varphi \to \psi)$; $\mathrm{ltl}(4,\varphi \to \psi) \coloneqq \mathrm{ltl}(4,\varphi) \to \mathrm{ltl}(4,\psi)$
Robust next	$\odot$	$1 \le i \le 4$: $\mathrm{ltl}(i,\odot\varphi) \coloneqq \bigcirc\mathrm{ltl}(i,\varphi)$
Robust eventually	$\diamonddot$	$1 \le i \le 4$: $\mathrm{ltl}(i,\diamonddot\varphi) \coloneqq \Diamond\mathrm{ltl}(i,\varphi)$
Robust always	$\boxdot$	$\mathrm{ltl}(1,\boxdot\varphi) \coloneqq \Box\mathrm{ltl}(1,\varphi)$; $\mathrm{ltl}(2,\boxdot\varphi) \coloneqq \Diamond\Box\mathrm{ltl}(2,\varphi)$; $\mathrm{ltl}(3,\boxdot\varphi) \coloneqq \Box\Diamond\mathrm{ltl}(3,\varphi)$; $\mathrm{ltl}(4,\boxdot\varphi) \coloneqq \Diamond\mathrm{ltl}(4,\varphi)$
Robust until	$\mathbf{U}$	$1 \le i \le 4$: $\mathrm{ltl}(i,\varphi\,\mathbf{U}\,\psi) \coloneqq \mathrm{ltl}(i,\varphi)\,\mathbf{U}\,\mathrm{ltl}(i,\psi)$
Robust release	$\mathbf{R}$	$\mathrm{ltl}(1,\varphi\,\mathbf{R}\,\psi) \coloneqq \mathrm{ltl}(1,\varphi)\,\mathbf{R}\,\mathrm{ltl}(1,\psi)$; $\mathrm{ltl}(2,\varphi\,\mathbf{R}\,\psi) \coloneqq \Diamond\Box\mathrm{ltl}(2,\psi) \vee \Diamond\mathrm{ltl}(2,\varphi)$; $\mathrm{ltl}(3,\varphi\,\mathbf{R}\,\psi) \coloneqq \Box\Diamond\mathrm{ltl}(3,\psi) \vee \Diamond\mathrm{ltl}(3,\varphi)$; $\mathrm{ltl}(4,\varphi\,\mathbf{R}\,\psi) \coloneqq \Diamond\mathrm{ltl}(4,\psi) \vee \Diamond\mathrm{ltl}(4,\varphi)$

at infinitely many positions of σ, and (d) $V_r(\sigma,\boxdot\varphi)[4] = 1$ if φ holds at some position of σ. For a thorough introduction and motivation, see the original work [50].

Finally, note that rLTL is an extension of LTL. In fact, the LTL semantics can be recovered from the first bit of the rLTL semantics [50] (after every implication $\varphi \to \psi$ has been replaced with $\neg\varphi \vee \psi$).[9]

3 Monitoring Robust LTL

In their work on LTL monitoring, Bauer et al. [14] define the problem of runtime monitoring as *"check[ing] LTL properties given finite prefixes of infinite [words]"*. More formally, given some prefix $u \in \Sigma^*$ and some LTL formula φ, it asks whether all, some, or no infinite extension $u\sigma \in \Sigma^\omega$ of u by some $\sigma \in \Sigma^\omega$ satisfies φ. To reflect these three possible results, the authors use the set $\mathbb{B}^? = \{0, ?, 1\}$ to define a three-valued logic that is syntactically identical to LTL, but equipped with a semantics in form of an evaluation function $V^m\colon \Sigma^* \times \Phi_{LTL} \to \mathbb{B}^?$ over finite prefixes. This semantics is defined such that $V^m(u,\varphi)$ is equal to 0 (is equal to 1) if no (if every) extension $u\sigma$ of u satisfies φ. If neither is the case, i.e., if there is an extension of u that satisfies φ and there is an extension of u that does not satisfy φ, then $V^m(u,\varphi)$ is equal to ?.

We aim to extend the approach of Bauer et al. to rLTL, whose semantics is based on truth values from the set $\mathbb{B}_4$ (containing the sequences of length four in 0^*1^*). As a motivating example, let us consider the formula $\varphi = \boxdot s$ for some atomic proposition s and study which situations can arise when monitoring this formula. The semantics of φ is given by concatenating the truth values of the LTL formulas $\mathrm{ltl}(1,\varphi) = \Box s$, $\mathrm{ltl}(2,\varphi) = \Diamond\Box s$, $\mathrm{ltl}(3,\varphi) = \Box\Diamond s$, and $\mathrm{ltl}(4,\varphi) = \Diamond s$.

First, consider the empty prefix and its two extensions $\emptyset^\omega$ and $\{s\}^\omega$. We have $V_r(\emptyset^\omega,\varphi) = 0000$ and $V_r(\{s\}^\omega,\varphi) = 1111$. Thus, all four bits can both be equal to 0 and 1. This situation will be captured by the sequence ???? which signifies that for every position i and every bit $b \in \mathbb{B}$, there exists an extension of ε that has bit b in the i-th position of the truth value with respect to φ.

Now, consider the prefix $\{s\}$ for which we have $V_r(\{s\}\sigma,\varphi)[4] = 1$ for every $\sigma \in \Sigma^\omega$ as $\mathrm{ltl}(4,\varphi)$ is satisfied on each extension of $\{s\}$ (s has already occurred). On the other hand, we have $V_r(\{s\}\emptyset^\omega,\varphi) = 0001$ and $V_r(\{s\}\{s\}^\omega,\varphi) = 1111$, i.e., the first three bits can both be 0 and 1 by picking an appropriate extension.

[9] It turns out that Tabuada and Neider's original proof [50, Proposition 5] has a minor mistake. Although the first bit of the rLTL semantics coincides with the original LTL semantics for all formulas that do not contain implications, the formula $\boxdot\neg a \to \boxdot a$ is an example witnessing this claim is no longer correct in the presence of implications. However, this issue can be fixed by replacing every implication $\varphi \to \psi$ with $\neg\varphi \vee \psi$. This substitution results in an equivalent LTL formula for which the first bit of the rLTL semantics indeed coincides with the LTL semantics.

Hence, the situation will be captured by the sequence ???1, signifying that the last bit is determined by the prefix, but the first three are not. Using dual arguments, the sequence 0??? is used for the prefix $\emptyset$, signifying that the first bit is determined by the prefix as every extension violates $\mathrm{ltl}(1, \varphi)$. However, the last three bits are not yet determined by the prefix, hence the trailing ?'s.

Finally, consider the prefix $\{s\}\emptyset$. Using the same arguments as for the previous two prefixes, we obtain $V_r(\{s\}\emptyset\sigma, \varphi)[1] = 0$ and $V_r(\{s\}\emptyset\sigma, \varphi)[4] = 1$ for every $\sigma \in \Sigma^\omega$. Also, as before, we have $V_r(\{s\}\emptyset\emptyset^\omega, \varphi) = 0001$ and $V_r(\{s\}\emptyset\{s\}^\omega, \varphi) = 0111$. Hence, here we obtain the sequence 0??1 signifying that the first and last bit are determined by the prefix, but the middle two are not.

In general, we use truth values of the form $0^*?^*1^*$, which follows from the fact that the truth values of rLTL are in 0^*1^*. Hence, let $\mathbb{B}_4^?$ denote the set of sequences of length four in $0^*?^*1^*$. Based on $\mathbb{B}_4^?$, we now formally define the rLTL monitoring semantics as a bitwise generalization of the LTL definition.

Definition 2. *The robust monitoring semantics $V_r^m \colon \Sigma^* \times \Phi_{rLTL} \to \mathbb{B}_4^?$ is defined as $V_r^m(u, \varphi) = \beta$ with*

$$\beta[i] = \begin{cases} 0 & \text{if } V_r(u\sigma, \varphi)[i] = 0 \text{ for all } \sigma \in \Sigma^\omega; \\ 1 & \text{if } V_r(u\sigma, \varphi)[i] = 1 \text{ for all } \sigma \in \Sigma^\omega; \text{ and} \\ ? & \text{otherwise,} \end{cases}$$

for every $i \in \{1, \ldots, 4\}$, every rLTL formula φ, and every $u \in \Sigma^*$.

Using this semantics, we are able to infer information about the infinite run of a system after having read only a finite prefix thereof. In fact, this robust semantics provides far more information about the degree of violation of the specification than classical LTL monitoring as each bit of the monitoring output represents a degree of violation of the specification: a ? turning into a 0 or 1 indicates a deterioration or improvement in the system's state, respectively. Consider, for instance, an autonomous drone with specification $\varphi = \boxdot s$ where s denotes a state of stable flight (recall the motivating example on Page 4). Initially, the monitor would output ???? due to a lack of information. If taking off under windy conditions, the state s is not reached initially, hence the monitor issues a warning by producing $V_r^m(\emptyset^n, \varphi) = 0???$ for every $n > 0$. Thus, the safety condition is violated temporarily, but not irrecoverably. Hence, mitigation measures can be initiated. Upon success, the monitoring output turns into $V_r^m(\emptyset^n\{s\}, \varphi) = 0??1$ for every $n > 0$, signaling that flight was stable for some time.

Before we continue, let us first state that the new semantics is well-defined, i.e., that the sequence $\beta[1]\beta[2]\beta[3]\beta[4]$ in Definition 2 is indeed in $\mathbb{B}_4^?$.

Lemma 1. *$V_r^m(u, \varphi) \in \mathbb{B}_4^?$ for every rLTL formula φ and every $u \in \Sigma^*$.*

After having shown that every possible output of V_r^m is in $\mathbb{B}_4^?$, the next obvious question is whether V_r^m is surjective, i.e., whether every truth value $\beta \in \mathbb{B}_4^?$ is *realized* by some prefix $u \in \Sigma^*$ and some rLTL formula φ in the sense that $V_r^m(u, \varphi) = \beta$. Recall the motivating example above: the formula $\boxdot s$ realizes at least the following four truth values: ???? (on ε), ???1 (on $\{s\}$), 0??? (on $\emptyset$), and 0??1 (on $\{s\}\emptyset$). It is not hard to convince oneself that these are all truth values realized by $\boxdot s$ as they represent the following four types of prefixes that can be distinguished: the prefix is empty (truth value ????), the prefix is in $\{s\}^+$ (truth value ???1), the prefix is in $\emptyset^+$ (truth value 0???), or the prefix contains both an $\{s\}$ and an $\emptyset$ (truth value 0??1).

For most other truth values, it is straightforward to come up with rLTL formulas and prefixes that realize them. See Table 2 for an overview.

For others, such as 0011, it is much harder. Intuitively, to realize 0011, one needs to find an rLTL formula φ and a prefix $u \in \Sigma^*$ such that the formula obtained by replacing all $\boxdot$ in φ by $\Diamond\Box$ is not satisfied by any extension of u, but the formula obtained by replacing all $\boxdot$ in φ by $\Box\Diamond$ is satisfied by every extension of u.[10] Thus, intuitively, the prefix has to differentiate between a property holding almost always and holding infinitely often. It turns out that no such u and φ exist. A similar argument is true for 0001, leading to the following theorem.

[10] Note that this intuition breaks down in the presence of implications and negation, due to their non-standard definitions.

Table 2. Realizable truth values. For every truth value β, the next two columns show prefixes u and formulas φ such that $V_r^m(u,\varphi)=\beta$, or that β is unrealizable.

Value	*Prefix*	*Formula*	*Value*	*Prefix*	*Formula*
0000	ε	$a \wedge \neg a$	0?11	$\emptyset\{a\}$	$\boxdot a \vee \boxdot \neg a$
000?	ε	$\Diamond\boxdot a \wedge \diamonddot \neg \diamonddot a$	0111	$\emptyset\{a\}$	$a \,\mathbf{R}\, a$
0001	unrealizable		????	ε	$\boxdot a$
00??	ε	$\boxdot a \wedge \boxdot \neg a$	???1	$\{a\}$	$\boxdot a$
00?1	$\emptyset\{a\}$	$\boxdot a \wedge \boxdot \neg a$	??11	ε	$\boxdot a \vee \diamonddot \neg \diamonddot a$
0011	unrealizable		?111	ε	$\boxdot a \vee \neg \diamonddot \neg \diamonddot \neg a$
0???	$\emptyset$	$\boxdot a$	1111	ε	$a \vee \neg a$
0??1	$\emptyset\{a\}$	$\boxdot a$			

Theorem 1. *All truth values except for* 0011 *and* 0001 *are realizable.*

As shown in Table 2, all of the realizable truth values except for 0111 are realized by formulas using only conjunction, disjunction, negation, eventually, and always. Further, 0111 can only be realized by a formula with the release operator while the truth values 0011 and 0001 are indeed not realizable at all.

Note that the two unrealizable truth values 0011 and 0001 both contain a 0 that is directly followed by a 1. The proof of unrealizability formalizes the intuition that such an "abrupt" transition from definitive violation of a property to definitive satisfaction of the property cannot be witnessed by any finite prefix. Finally, the only other truth value of this form, 0111, is only realizable by using a formula with the release operator.

Going again back to the motivating example $\boxdot s$, consider the evolution of the truth values on the sequence $\varepsilon, \{s\}, \{s\}\emptyset$: They are ????, ???1, and 0??1, i.e., 0's and 1's are stable when extending a prefix, only a ? may be replaced by a 0 or a 1. This property holds in general. To formalize this, say that $\beta' \in \mathbb{B}_4^?$ is more specific than $\beta \in \mathbb{B}_4^?$, written as $\beta \preceq \beta'$, if, for all i, $\beta[i] \neq ?$ implies $\beta'[i] = \beta[i]$.

Lemma 2. *Let φ be an rLTL formula and $u, u' \in \Sigma^*$. If $u \sqsubseteq u'$, then $V_r^m(u,\varphi) \preceq V_r^m(u',\varphi)$.*

Lemma 2 immediately implies two properties of the semantics: *impartiality* and *anticipation* [18]. Impartiality states that a definitive verdict will never be revoked: if $V_r^m(u,\varphi)[i] \neq ?$, then for all finite extensions $v \in \Sigma^*$, the verdict will not change, so $V_r^m(uv,\varphi)[i] = V_r^m(u,\varphi)[i]$. Anticipation requires that a definitive verdict is decided as soon as possible, i.e., if $V_r^m(u,\varphi)[i] = ?$, then u can still be extended to satisfy and to violate $\mathrm{ltl}(i,\varphi)$. This holds by definition of $V_r^m(u,\varphi)$.

Due to Lemma 2, for a fixed formula, the prefixes of every infinite word can assume at most five different truth values, which are all of increasing specificity. It turns out that this upper bound is tight. To formalize this claim, we denote the strict version of $\preceq$ by $\prec$, i.e., $\beta \prec \beta'$ if and only if $\beta \preceq \beta'$ and $\beta \neq \beta'$.

Lemma 3. *There is an rLTL formula φ and prefixes $u_0 \sqsubset u_1 \sqsubset u_2 \sqsubset u_3 \sqsubset u_4$ such that $V_r^m(u_0,\varphi) \prec V_r^m(u_1,\varphi) \prec V_r^m(u_2,\varphi) \prec V_r^m(u_3,\varphi) \prec V_r^m(u_4,\varphi)$.*

Finally, let us consider the notion of *monitorability* [47], an important concept in the theory of runtime monitoring. As a motivation, consider the LTL formula $\psi = \Box\Diamond s$ and an arbitrary prefix $u \in \Sigma^*$. Then, the extension $u\{s\}^\omega$ satisfies ψ while the extension $u\emptyset^\omega$ does not satisfy ψ, i.e., satisfaction of ψ is independent of any prefix u. Hence, we have $V^m(u,\psi) = ?$ for every prefix u, i.e., monitoring the formula ψ does not generate any information.

In general, for a fixed LTL formula φ, a prefix $u \in \Sigma^*$ is called *ugly* if we have $V^m(uv,\varphi) = ?$ for every finite $v \in \Sigma^*$, i.e., every finite extension of u yields an indefinite verdict. [11] Now, φ is *LTL-monitorable* if there is no ugly prefix with respect to φ. A wide range of LTL formulas, e.g., $\psi = \Box\Diamond s$ as above, are

[11] Ugly prefixes [14] complement the good and bad prefixes of Kuperman and Vardi [37].

unmonitorable in that sense. In particular, 44% of the LTL formulas considered in the experiments of Bauer et al. are not LTL-monitorable.

We next generalize the notion of monitorability to rLTL. In particular, we answer whether there are unmonitorable rLTL formulas. Then, in Section 5, we exhibit that all LTL formulas considered by Bauer et al.'s experimental evaluation, even the unmonitorable ones, are monitorable under rLTL semantics. To conclude the motivating example, note that the rLTL analogue $\boxdot\Diamond s$ of the LTL formula ψ induces two truth values from $\mathbb{B}_4^?$ indicating whether s has been true at least once (truth value ???1) or not (truth value ????). Even more so, every prefix inducing the truth value ???? can be extended to one inducing the truth value ???1.

Definition 3. *Let φ be an rLTL formula. A prefix $u \in \Sigma^*$ is called* ugly *if we have $V^m(uv, \varphi) = ????$ for every finite $v \in \Sigma^*$. Further, φ is* rLTL-monitorable *if it has no ugly prefix.*

As argued above, the formula $\boxdot\Diamond s$ has no ugly prefix, i.e., it is rLTL-monitorable. Thus, we have found an unmonitorable LTL formula whose rLTL analogue (the formula obtained by adding dots to all temporal operators) is monitorable. The converse statement is also true. There is a monitorable LTL formula whose rLTL analogue is unmonitorable. To this end, consider the formula

$$(\Box s \wedge \Box \neg s) \rightarrow (\Diamond \Box s \wedge \Diamond \neg \Diamond s),$$

which is a tautology and therefore monitorable. On the other hand, we claim that $\emptyset\{s\}$ is an ugly prefix for the rLTL analogue φ obtained by adding dots to the temporal operators. To this end note that we have $V_r(\emptyset\{s\}v\emptyset^\omega, \varphi) = 1111$ for every $v \in \Sigma^*$ and $V_r(\emptyset\{s\}v\{s\}^\omega, \varphi) = 0000$ for every $v \in \Sigma^*$. Hence, $V_r^m(\emptyset\{s\}v, \varphi) = ????$ for every such v, i.e., $\emptyset\{s\}$ is indeed ugly and φ therefore not rLTL-monitorable.

As a final example, consider the negation $\neg\boxdot\Diamond s$ of the above rLTL-monitorable formula. It is not hard to see that we have $V_r^m(u, \varphi) = ????$ for every prefix u. Hence, ε is an ugly prefix for the formula, i.e., we have found another unmonitorable rLTL formula. In particular, the example shows that, unlike for LTL, rLTL-monitorability is not preserved under negation.

In the next section, we show our main result: The robust monitoring semantics V_r^m can be implemented by finite-state machines.

4 Construction of rLTL Monitors

An rLTL monitor is an implementation of the robust monitoring semantics V_r^m in form of a finite-state machine with output. More precisely, an *rLTL monitor* for an rLTL formula φ is a finite state-machine $\mathcal{M}_\varphi$ that on reading an input $u \in \Sigma^*$ outputs $V_r^m(u, \varphi)$. In this section, we show how to construct rLTL monitors and that this construction is asymptotically not more expensive than the construction of LTL monitors. Let us fix an rLTL formula φ for the remainder of this section.

Our construction of rLTL monitors is inspired by Bauer et al. [14] and generates a sequence of finite-state machines (i.e., Büchi automata over infinite words, (non)deterministic automata over finite words, and Moore machines). Underlying these machines are *transition structures* $\mathcal{T} = (Q, q_I, \Delta)$ consisting of a nonempty, finite set Q of states, an initial state $q_I \in Q$, and a transition relation $\Delta \subseteq Q \times \Sigma \times Q$. An (infinite) run of $\mathcal{T}$ on a word $\sigma = a_0a_1a_2 \cdots \in \Sigma^\omega$ is a sequence $\rho = q_0q_1 \cdots$ of states such that $q_0 = q_I$ and $(q_j, a_j, q_{j+1}) \in \Delta$ for $j \in \mathbb{N}$. Finite runs on finite words are defined analogously. The transition structure $\mathcal{T}$ is *deterministic* if (a) $(q, a, q') \in \Delta$ and $(q, a, q'') \in \Delta$ imply $q' = q''$ and (b) for each $q \in Q$ and $a \in \Sigma$ there exists a $(q, a, q') \in \Delta$. We then replace the transition relation Δ by a function $\delta\colon Q \times \Sigma \rightarrow Q$. Finally, we define the *size* of a transition structure $\mathcal{T}$ as $|\mathcal{T}| = |Q|$ in order to measure its complexity.

Our construction then proceeds in three steps:

1. We bring φ into an operational form by constructing Büchi automata $\mathcal{A}_\beta^\varphi$ for each truth value $\beta \in \mathbb{B}_4$ that can decide the valuation $V_r(\sigma, \varphi)$ of infinite words $\sigma \in \Sigma^\omega$.
2. Based on these Büchi automata, we then construct nondeterministic automata $\mathcal{B}_\beta^\varphi$ that can decide whether a finite word $u \in \Sigma^*$ can still be extended to an infinite word $u\sigma \in \Sigma^\omega$ with $V_r(u\sigma, \varphi) = \beta$.

3. We determinize the nondeterministic automata obtained in Step 2 and combine them into a single Moore machine that computes $V_r^m(u,\varphi)$.

Let us now describe each of these steps in detail.

Step 1: We first translate the rLTL formula φ into several Büchi automata using a construction by Tabuada and Neider [50], summarized in Theorem 2 below. A Büchi automaton is a four-tuple $\mathcal{A} = (Q, q_I, \Delta, F)$ where $\mathcal{T} = (Q, q_I, \Delta)$ is a transition structure and $F \subseteq Q$ is a set of accepting states. A run π of $\mathcal{A}$ on $\sigma \in \Sigma^\omega$ is a run of $\mathcal{T}$ on σ, and we say that π is accepting if it contains infinitely many states from F. The automaton $\mathcal{A}$ accepts a word σ if there exists an accepting run of $\mathcal{A}$ on σ. The language $\mathcal{L}(\mathcal{A})$ is the set of all words accepted by $\mathcal{A}$, and the size of $\mathcal{A}$ is defined as $|\mathcal{A}| = |\mathcal{T}|$.

Theorem 2 (Tabuada and Neider [50]). *Given a truth value $\beta \in \mathbb{B}_4$, one can construct a Büchi automaton $\mathcal{A}_\beta^\varphi$ with $2^{\mathcal{O}(|\varphi|)}$ states such that $\mathcal{L}(\mathcal{A}_\beta^\varphi) = \{\sigma \in \Sigma^\omega \mid V_r(\sigma,\varphi) = \beta\}$. This construction can be performed in $2^{\mathcal{O}(|\varphi|)}$ time.*

The Büchi automata $\mathcal{A}_\beta^\varphi$ for $\beta \in \mathbb{B}_4$ serve as building blocks for the next steps.

Step 2: For each Büchi automaton $\mathcal{A}_\beta^\varphi$ obtained in the previous step, we now construct a nondeterministic automaton $\mathcal{B}_\beta^\varphi$ over finite words. This automaton determines whether a finite word $u \in \Sigma^*$ can be continued to an infinite word $u\sigma \in \mathcal{L}(\mathcal{A}_\beta^\varphi)$ (i.e., $V_r(u\sigma,\varphi) = \beta$) and is used later to construct the rLTL monitor.

Formally, a *nondeterministic finite automaton (NFA)* is a four-tuple $\mathcal{A} = (Q, q_I, \Delta, F)$ that is syntactically identical to a Büchi automaton. The size of $\mathcal{A}$ as well is is defined analogously to Büchi automata. In contrast to Büchi automata, however, NFAs only admit finite runs on finite words, i.e., a run of $\mathcal{A}$ on $u = a_0 \cdots a_{n-1} \in \Sigma^*$ is a sequence $q_0 \cdots q_n$ such that $q_0 = q_I$ and $(q_j, a_j, q_{j+1}) \in \Delta$ for every $j < n$. A run $q_0 \cdots q_n$ is called *accepting* if $q_n \in F$. Accepted words as well as the language of $\mathcal{A}$ are again defined analogously to the Büchi case. If (Q, q_I, Δ) is deterministic, $\mathcal{A}$ is a *deterministic finite automaton (DFA)*. It is well-known that for each NFA $\mathcal{A}$ one can construct a DFA $\mathcal{A}'$ with $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}')$ and $|\mathcal{A}'| \in \mathcal{O}(2^{|\mathcal{A}|})$.

Given the Büchi automaton $\mathcal{A}_\beta^\varphi = (Q_\beta, q_{I,\beta}, \Delta_\beta, F_\beta)$, we first compute the set $F_\beta^\star = \{q \in Q_\beta \mid \mathcal{L}(\mathcal{A}_\beta^\varphi(q)) \neq \emptyset\}$, where $\mathcal{A}_\beta^\varphi(q)$ denotes the Büchi automaton $\mathcal{A}_\beta^\varphi$ but with initial state q instead of q_I. Intuitively, the set $F_\beta^\star$ contains all states $q \in Q_\beta$ from which there exists an accepting run in $\mathcal{A}_\beta^\varphi$ and, hence, indicates whether a finite word $u \in \Sigma^*$ reaching a state of $F_\beta^\star$ can be extended to an infinite word $u\sigma' \in \mathcal{L}(\mathcal{A}_\beta^\varphi)$. The set $F_\beta^\star$ can be computed, for instance, using a nested depth-first search [49] for each state $q \in Q_\beta$. Since each such search requires time quadratic in $|\mathcal{A}_\beta^\varphi|$, the set $F_\beta^\star$ can be computed in time $\mathcal{O}(|\mathcal{A}_\beta^\varphi(q)|^3)$.

Using $F_\beta^\star$, we define the NFA $\mathcal{B}_\beta^\varphi = (Q_\beta, q_{I,\beta}, \Delta_\beta, F_\beta^\star)$ that shares the transition structure of $\mathcal{A}_\beta^\varphi$ and uses $F_\beta^\star$ as the set of final states. The next lemma states that $\mathcal{B}_\beta^\varphi$ indeed recognizes prefixes of words in $\mathcal{L}(\mathcal{A}_\beta^\varphi)$.

Lemma 4. *Let $\beta \in \mathbb{B}_4$ and $u \in \Sigma^*$. Then, $u \in \mathcal{L}(\mathcal{B}_\beta^\varphi)$ if and only if there exists an infinite word $\sigma \in \Sigma^\omega$ with $V_r(u\sigma,\varphi) = \beta$.*

Before we continue to the last step in our construction, let us briefly comment on the complexity of computing the NFAs $\mathcal{B}_\beta^\varphi$. Since $\mathcal{B}_\beta^\varphi$ and $\mathcal{A}_\beta^\varphi$ share the same underlying transition system, we immediately obtain $|\mathcal{B}_\beta^\varphi| \in 2^{\mathcal{O}(|\varphi|)}$. Moreover, the construction of $\mathcal{B}_\beta^\varphi$ is dominated by the computation of the set $F_\beta^\star$ and, hence, can be done in time $2^{\mathcal{O}(|\varphi|)}$.

Step 3: In the final step, we construct a Moore machine implementing an rLTL monitor for φ. Formally, a *Moore machine* is a five-tuple $\mathcal{M} = (Q, q_I, \delta, \Gamma, \lambda)$ consisting of a deterministic transition system (Q, q_I, δ), an output alphabet Γ, and an output function $\lambda \colon Q \to \Gamma$. The size of $\mathcal{M}$ as well of runs of $\mathcal{M}$ are defined as for DFAs. In contrast to a DFA, however, a Moore machine $\mathcal{M}$ computes a function $\lambda_\mathcal{M} \colon \Sigma^* \to \Gamma$ that is defined by $\lambda_\mathcal{M}(u) = \lambda(q_n)$ where q_n is the last state reached on the unique finite run $q_0 \cdots q_n$ of $\mathcal{M}$ on its input $u \in \Sigma^*$.

The first step in the construction of the Moore machine is to determinize the NFAs $\mathcal{B}^\varphi_\beta$, obtaining equivalent DFAs $\mathcal{C}^\varphi_\beta = (Q'_\beta, q'_{I,\beta}, \delta'_\beta, F'_\beta)$ of at most exponential size in $|\mathcal{B}^\varphi_\beta|$. Subsequently, we combine these DFAs into a single Moore machine $\mathcal{M}_\varphi$ implementing the desired rLTL monitor. Intuitively, this Moore machine is the product of the DFAs $\mathcal{C}^\varphi_\beta$ for each $\beta \in \mathbb{B}_4$ and tracks the run of each individual DFA on the given input. Formally, $\mathcal{M}_\varphi$ is defined as follows.

Definition 4. *Let $\mathbb{B}_4 = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5\}$. We define $\mathcal{M}_\varphi = (Q, q_I, \Gamma, \delta, \lambda)$ by*

- $Q = Q'_{\beta_1} \times Q'_{\beta_2} \times Q'_{\beta_3} \times Q'_{\beta_4} \times Q'_{\beta_5}$;
- $q_I = (q'_{I,\beta_1}, q'_{I,\beta_2}, q'_{I,\beta_3}, q'_{I,\beta_4}, q'_{I,\beta_5})$;
- $\delta\big((q_1, q_2, q_3, q_4, q_5), a\big) = (q'_1, q'_2, q'_3, q'_4, q'_5)$ *where* $q'_j = \delta'_{\beta_j}(q_j, a)$ *for each* $j \in \{1, \dots, 5\}$;
- $\Gamma = \mathbb{B}_4^?$; *and*
- $\lambda\big((q_1, q_2, q_3, q_4, q_5)\big) = \xi\big(\{\beta_j \in \mathbb{B}_4 \mid q_j \in F'_{\beta_j}, j \in \{1, \dots, 5\}\}\big)$,

where the surjective function $\xi \colon 2^{\mathbb{B}_4} \to \mathbb{B}_4^?$ translates sets $B \subseteq \mathbb{B}_4$ of truth values to the robust monitoring semantics as follows: $\xi(B) = \beta^? \in \mathbb{B}_4^?$ with

$$\beta^?[j] = \begin{cases} 0 & \textit{if } \beta[j] = 0 \textit{ for each } \beta \in B; \\ 1 & \textit{if } \beta[j] = 1 \textit{ for each } \beta \in B; \textit{ and} \\ ? & \textit{otherwise.} \end{cases}$$

The main result of this paper now shows that the Moore machine $\mathcal{M}_\varphi$ implements V_r^m, i.e., we have $\lambda_{\mathcal{M}_\varphi}(u) = V_r^m(u, \varphi)$ for every prefix u.

Theorem 3. *For every rLTL formula φ, one can construct an rLTL monitor of size $2^{2^{\mathcal{O}(|\varphi|)}}$.*

In a final post-processing step, we minimize $\mathcal{M}_\varphi$ (e.g., using one of the standard algorithms for deterministic automata). As a result, we obtain the unique minimal monitor for the given rLTL formula.

It is left to determine the complexity of our rLTL monitor construction. Since each DFA $\mathcal{C}^\varphi_\beta$ is in the worst case exponential in the size of the NFA $\mathcal{B}^\varphi_\beta$, we immediately obtain that $\mathcal{C}^\varphi_\beta$ is at most of size $2^{2^{\mathcal{O}(|\varphi|)}}$. Thus, the Moore machine $\mathcal{M}_\varphi$ is at most of size $2^{2^{\mathcal{O}(|\varphi|)}}$ as well and can be effectively computed in doubly-exponential time in $|\varphi|$. Note that this matches the complexity bound of Bauer et al.'s approach for LTL runtime monitoring [14]. Moreover, this bound is tight since rLTL subsumes LTL and a doubly-exponential bound is tight for LTL [37,14]. Hence, robust runtime monitoring asymptotically incurs no extra cost compared to classical LTL runtime monitoring. However, it provides more useful information as we demonstrate next in our experimental evaluation.

5 Experimental Evaluation

Besides incorporating a notion of robustness into classical LTL monitoring, our rLTL monitoring approach also promises to provide richer information than its LTL counterpart. In this section, we evaluate empirically whether this promise is actually fulfilled. More precisely, we answer the following two questions on a comprehensive suite of benchmarks:

1. How does rLTL monitoring compare to classical LTL monitoring in terms of monitorability?
2. For formulas that are both LTL-monitorable and rLTL-monitorable, how do both approaches compare in terms of the size of the resulting monitors and the time required to construct them?

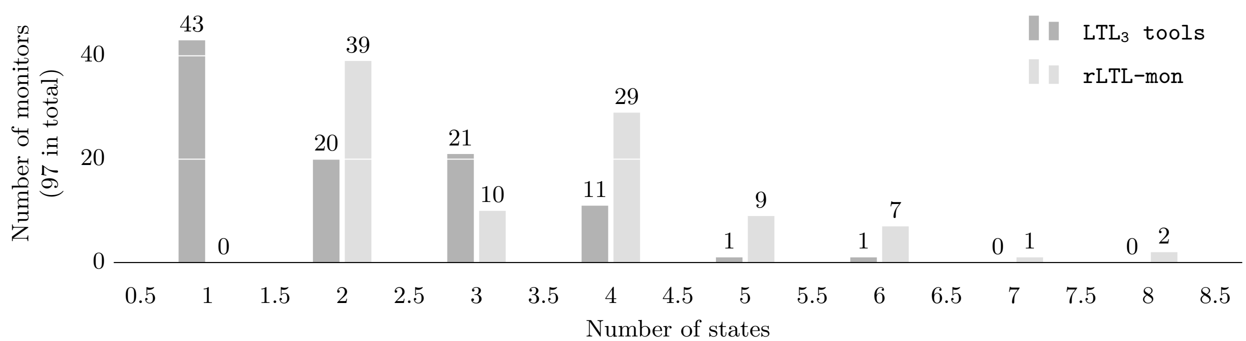
To answer these research questions, we have implemented a prototype, which we named `rLTL-mon`. Our prototype is written in Java and builds on top of two libraries: Owl [36], a library for LTL and automata over infinite words, as well as AutomataLib (part of LearnLib [34]), a library for automata over finite words and Moore machines. For technical reasons (partly due to limitations of the Owl library and partly to simplify the implementation), `rLTL-mon` uses a monitor construction that is slightly different from the one described in Section 4: instead of translating an rLTL formula into nondeterministic Büchi automata, `rLTL-mon` constructs deterministic parity automata. These parity automata are then directly converted into DFAs, thus skirting the need for a detour over NFAs and a subsequent determinization step. Note, however, that this alternative construction produces the same rLTL monitors than the one described in Section 4. Moreover, it has the same asymptotic complexity. The sources of our prototype are available online under an open-source license.[12]

Benchmarks and Experimental Setup Starting point of our evaluation was the original benchmark suite of Bauer et al. [14], which is based on a survey by Dwyer on frequently used software specification patterns [22]. This benchmark suite consists of 97 LTL formulas and covers a wide range of patterns, including safety, scoping, precedence, and response patterns. For our rLTL monitor construction, we interpreted each LTL formula in the benchmark suite as an rLTL formula (by treating every operator as a robust operator).

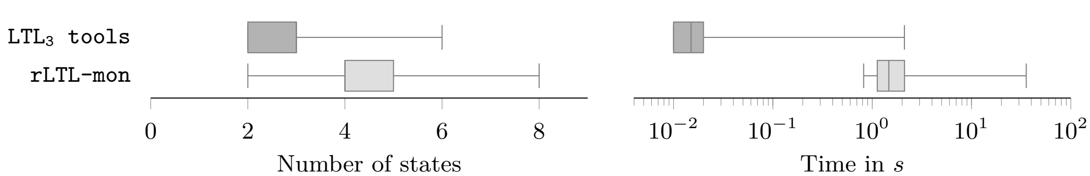
We have compared `rLTL-mon` to Bauer et al.'s implementation of their LTL monitoring approach, which the authors named $\texttt{LTL}_3$ `tools`. This tool uses LTL2BA [29] to translate LTL formulas into Büchi automata and AT&T's fsmlib as a means to manipulate finite-state machines. Since LTL2BA's and Owl's input format for LTL formulas do not match exactly, we have translated all benchmarks into a suitable format using a python script.

We conducted all experiments on an Intel Core i5-6600 @ 3.3 GHz inside a virtual machine with 4 GB of RAM running Ubuntu 18.04 LTS. As no monitor construction took longer than 600 s, we did not impose any time limit.

Results Our evaluation shows that both $\texttt{LTL}_3$ `tools` and `rLTL-mon` are able to generate monitors for all 97 formulas in Bauer et al.'s benchmark suite. Aggregated statistics of this evaluation are visualized in Figure 1.[13]



(a) Histogram of the number of monitors with respect to their size



(b) Analysis of the monitor construction for the 54 formulas that are both LTL-monitorable and rLTL-monitorable

Fig. 1. Comparison of `rLTL-mon` and $\texttt{LTL}_3$ `tools` on Bauer et al.'s benchmarks [14]

[12] http://bit.ly/rltl-sources

[13] Detailed results can be found online at http://bit.ly/rltl-monitoring-evaluation.

The histogram in Figure 1a shows the aggregate number of LTL and rLTL monitors with respect to their number of states. As Bauer et al. already noted in their original work, the resulting LTL monitors are quite small (none had more than six states), which they attribute to Dwyer et al.'s specific selection of formulas [22]. A similar observation is also true for the rLTL monitors: none had more than eight states.

In order to determine which formulas are monitorable and which are not, we used a different, though equivalent definition, which is easy to check on the monitor itself: an LTL formula (rLTL formula) is monitorable if and only if the unique minimized LTL monitor (rLTL monitor) does not contain a sink-state with universal self-loop that outputs "?" (that outputs "????"). Bauer et al. report that 44.3 % of all LTL monitors (43 out of 97) have this property (in fact, exactly the 43 LTL monitors with a single state), which means that 44.3 % of all formulas in their benchmark suite are not LTL-monitorable. By contrast, all these formulas are rLTL-monitorable. Moreover, in 78.4 % of the cases (76 out of 97), the rLTL monitor has more distinct outputs than the LTL monitor, indicating that the rLTL monitor provides more fine-grained information of the property being monitored; in the remaining 21.6 %, both monitors have the same number of distinct outputs. These results answer our first research question strongly in favor of rLTL monitoring: *rLTL monitoring did in fact provide more information than its classical LTL counterpart. In particular, only* 55.7 % *of the benchmarks are LTL-monitorable, whereas* 100 % *are rLTL-monitorable.*

Let us now turn to our second research question and compare both monitoring approaches on the 54 formulas that are both LTL-monitorable and rLTL-monitorable. For these formulas, Figure 1b further provides statistical analysis of the generated monitors in terms of their size (left diagram) as well as the time required to generate them (right diagram). Each box in the diagrams shows the lower and upper quartile (left and right border of the box, respectively), the median (line within the box), and minimum and maximum (left and right whisker, respectively).

Let us first consider the size of the monitors (left diagram of Figure 1b). The majority of LTL monitors (52) has between two and four states, while the majority of rLTL monitors (45) has between two and five states. For 21 benchmarks, the LTL and rLTL monitors are of equal size, while the rLTL monitor is larger for the remaining 33 benchmarks (in no case is the LTL monitor larger than the rLTL monitor). On average, rLTL monitors are about 1.5 times larger than the corresponding LTL monitors.

Let us now discuss the time taken to construct the monitors. As the diagram on the right-hand-side of Figure 1b shows, `LTL3 tools` was considerably faster than `rLTL-mon` on a majority of benchmarks (around 0.1 s and 2.6 s per benchmark, respectively). For all 54 benchmarks, the rLTL monitor construction took longer than the construction of the corresponding LTL monitor (although there are two non-LTL-monitorable formulas for which the construction of the rLTL monitor was faster). However, we attribute this large runtime gap partly to the overhead caused by repeatedly starting the Java virtual machine, which is not required in the case of `LTL3 tools`. Note that this is not a concern in practice as a monitor is only constructed once before it is deployed.

Finally, our analysis answers our second question: *rLTL monitors are only slightly larger than the corresponding LTL monitors and although they require considerably more time to construct, the overall construction time was negligible for almost all benchmarks.*

6 Conclusion

In this paper we proposed a monitoring framework for rLTL [50]. While LTL only reports if a property is violated or satisfied, rLTL measures the degree of violation thereby granting more insight into a system. The semantics of rLTL are based on infinite words. In runtime monitoring, however, only a finite prefix of a system's run is available. As a result of the lack of information about the future, a full verdict is not necessarily possible. We thus adapted the semantics of rLTL to account for such prefixes by providing an output of four three-valued bits where a ? indicates a lack of information, similar to Bauer et al. [14]. The finite rLTL semantics can be realized as a finite state machine. The construction process is asymptotically no more expensive than the respective construction of an LTL monitor whereas the rLTL monitor always provides as least as much information as the LTL monitor.

In addition to this theoretical validation, we replicated Bauer et al.'s benchmark consisting of 97 LTL formulas compiled by Dwyer et al. [22] for an empirical analysis. With this, we are able to confirm our hypothesis that the rLTL monitor is strictly more informative than its LTL counterpart in 77% of their formulas. Moreover, Pnueli and Zaks [47] define that a formula is *monitorable* if the monitor will eventually reach a definitive verdict for any prefix. However, 44% of the formulas in their benchmark are unmonitorable. We transferred the definition to rLTL and observed that all of these formulas are rLTL-monitorable.

All in all, we can answer both research questions positively: monitoring a robust logic such as rLTL does provide more information than an LTL monitor and the construction process is still practical as evidenced by the same asymptotic complexity and comparable running time on Bauer et al.'s collection of formulas.

Bauer et al. [12] also introduced a four-valued semantics for LTL consisting of the usual satisfaction and violation, as well as $\top^p$ and $\bot^p$. An output of $\bot^p$ ($\top^p$) indicates that if the run were to end in the current state, the formula would be violated (satisfied). Since this semantics also works on prefixes, an extension incorporating the four-valued semantics on a bit-by-bit basis suggests itself.

Moreover, desired properties for cyber-physical systems often include real-time components such as "touch the ground at most 15 seconds after receiving a landing command". Monitors for logics taking real-time into account [15] such as STL [39,40] induce high computational overhead at runtime when compared to LTL and rLTL monitors. Thus, a real-time extension for rLTL retaining its low runtime cost would greatly increase its viability as specification language.

References

1. Abbas, H., Pant, Y.V., Mangharam, R.: Temporal logic robustness for general signal classes. In: Ozay and Prabhakar [45], pp. 45–56. https://doi.org/10.1145/3302504.3311817
2. Abbas, H., Rodionova, A., Bartocci, E., Smolka, S.A., Grosu, R.: Quantitative regular expressions for arrhythmia detection algorithms. In: Feret, J., Koeppl, H. (eds.) CMSB 2017. LNCS, vol. 10545, pp. 23–39. Springer (2017). https://doi.org/10.1007/978-3-319-67471-1_2
3. Adolf, F., Faymonville, P., Finkbeiner, B., Schirmer, S., Torens, C.: Stream runtime monitoring on UAS. In: Lahiri, S.K., Reger, G. (eds.) RV 2017. LNCS, vol. 10548, pp. 33–49. Springer (2017). https://doi.org/10.1007/978-3-319-67531-2_3
4. Alur, R., Fisman, D., Raghothaman, M.: Regular programming for quantitative properties of data streams. In: Thiemann, P. (ed.) ESOP 2016. LNCS, vol. 9632, pp. 15–40. Springer (2016). https://doi.org/10.1007/978-3-662-49498-1_2
5. Anevlavis, T., Neider, D., Phillipe, M., Tabuada, P.: Evrostos: the rLTL verifier. In: Ozay and Prabhakar [45], pp. 218–223. https://doi.org/10.1145/3302504.3311812
6. Anevlavis, T., Philippe, M., Neider, D., Tabuada, P.: Verifying rLTL formulas: now faster than ever before! In: CDC 2018. pp. 1556–1561. IEEE (2018). https://doi.org/10.1109/CDC.2018.8619014
7. Baier, C., Katoen, J.: Principles of model checking. MIT Press (2008)
8. Barringer, H., Falcone, Y., Havelund, K., Reger, G., Rydeheard, D.E.: Quantified event automata: Towards expressive and efficient runtime monitors. In: Giannakopoulou, D., Méry, D. (eds.) FM 2012. LNCS, vol. 7436, pp. 68–84. Springer (2012). https://doi.org/10.1007/978-3-642-32759-9_9
9. Barringer, H., Goldberg, A., Havelund, K., Sen, K.: Rule-based runtime verification. In: Steffen, B., Levi, G. (eds.) VMCAI 2004. LNCS, vol. 2937, pp. 44–57. Springer (2004). https://doi.org/10.1007/978-3-540-24622-0_5
10. Barringer, H., Rydeheard, D.E., Havelund, K.: Rule systems for run-time monitoring: from Eagle to RuleR. J. Log. Comput. **20**(3), 675–706 (2010). https://doi.org/10.1093/logcom/exn076
11. Basin, D.A., Harvan, M., Klaedtke, F., Zalinescu, E.: MONPOLY: monitoring usage-control policies. In: Khurshid, S., Sen, K. (eds.) RV 2011. LNCS, vol. 7186, pp. 360–364. Springer (2011). https://doi.org/10.1007/978-3-642-29860-8_27
12. Bauer, A., Leucker, M., Schallhart, C.: The good, the bad, and the ugly, but how ugly is ugly? In: Sokolsky, O., Tasiran, S. (eds.) RV 2007. LNCS, vol. 4839, pp. 126–138. Springer (2007). https://doi.org/10.1007/978-3-540-77395-5_11
13. Bauer, A., Leucker, M., Schallhart, C.: Comparing LTL semantics for runtime verification. J. Log. Comput. **20**(3), 651–674 (2010). https://doi.org/10.1093/logcom/exn075
14. Bauer, A., Leucker, M., Schallhart, C.: Runtime verification for LTL and TLTL. ACM Trans. Softw. Eng. Methodol. **20**(4), 14:1–14:64 (2011). https://doi.org/10.1145/2000799.2000800

15. Bernstein, A.J., Jr., P.K.H.: Proving real-time properties of programs with temporal logic. In: Howard, J., Reed, D.P. (eds.) SOSP 1981. pp. 1–11. ACM (1981). https://doi.org/10.1145/800216.806585
16. Caspi, P., Pilaud, D., Halbwachs, N., Plaice, J.: Lustre: A declarative language for programming synchronous systems. In: POPL 1987. pp. 178–188. ACM Press (1987). https://doi.org/10.1145/41625.41641
17. D'Angelo, B., Sankaranarayanan, S., Sánchez, C., Robinson, W., Finkbeiner, B., Sipma, H.B., Mehrotra, S., Manna, Z.: LOLA: runtime monitoring of synchronous systems. In: (TIME 2005). pp. 166–174. IEEE Computer Society (2005). https://doi.org/10.1109/TIME.2005.26
18. Decker, N., Leucker, M., Thoma, D.: Impartiality and anticipation for monitoring of visibly context-free properties. In: Legay, A., Bensalem, S. (eds.) RV 2013. LNCS, vol. 8174, pp. 183–200. Springer (2013). https://doi.org/10.1007/978-3-642-40787-1_11
19. Decker, N., Leucker, M., Thoma, D.: Monitoring modulo theories. STTT **18**(2), 205–225 (2016). https://doi.org/10.1007/s10009-015-0380-3
20. Donzé, A., Ferrère, T., Maler, O.: Efficient robust monitoring for STL. In: Sharygina, N., Veith, H. (eds.) CAV 2013. LNCS, vol. 8044, pp. 264–279. Springer (2013). https://doi.org/10.1007/978-3-642-39799-8_19
21. Drusinsky, D.: The temporal rover and the ATG rover. In: Havelund, K., Penix, J., Visser, W. (eds.) SPIN 2000. LNCS, vol. 1885, pp. 323–330. Springer (2000). https://doi.org/10.1007/10722468_19
22. Dwyer, M.B., Avrunin, G.S., Corbett, J.C.: Patterns in property specifications for finite-state verification. In: Boehm, B.W., Garlan, D., Kramer, J. (eds.) ICSE 1999. pp. 411–420. ACM (1999). https://doi.org/10.1145/302405.302672
23. Eisner, C., Fisman, D., Havlicek, J., Lustig, Y., McIsaac, A., Campenhout, D.V.: Reasoning with temporal logic on truncated paths. In: Hunt, W.A., Somenzi, F. (eds.) CAV 2003. LNCS, vol. 2725, pp. 27–39. Springer (2003). https://doi.org/10.1007/978-3-540-45069-6_3
24. Falcone, Y., Sánchez, C. (eds.): RV 2016, LNCS, vol. 10012. Springer (2016). https://doi.org/10.1007/978-3-319-46982-9
25. Faymonville, P., Finkbeiner, B., Schledjewski, M., Schwenger, M., Tentrup, L., Torfah, H.: Streamlab: Stream-based monitoring of cyber-physical systems. In: CAV 2019 (2019), to appear.
26. Faymonville, P., Finkbeiner, B., Schirmer, S., Torfah, H.: A stream-based specification language for network monitoring. In: Falcone and Sánchez [24], pp. 152–168. https://doi.org/10.1007/978-3-319-46982-9_10
27. Ferrère, T., Nickovic, D., Donzé, A., Ito, H., Kapinski, J.: Interface-aware signal temporal logic. In: Ozay and Prabhakar [45], pp. 57–66. https://doi.org/10.1145/3302504.3311800
28. Finkbeiner, B., Sankaranarayanan, S., Sipma, H.: Collecting statistics over runtime executions. Form. Meth. in Sys. Des. **27**(3), 253–274 (2005). https://doi.org/10.1007/s10703-005-3399-3
29. Gastin, P., Oddoux, D.: Fast LTL to Büchi automata translation. In: Berry, G., Comon, H., Finkel, A. (eds.) CAV 2001. LNCS, vol. 2102, pp. 53–65. Springer (2001). https://doi.org/10.1007/3-540-44585-4_6
30. Hallé, S.: When RV meets CEP. In: Falcone and Sánchez [24], pp. 68–91. https://doi.org/10.1007/978-3-319-46982-9_6
31. Havelund, K., Peled, D., Ulus, D.: First order temporal logic monitoring with BDDs. In: Stewart, D., Weissenbacher, G. (eds.) FMCAD 2017. pp. 116–123. IEEE (2017). https://doi.org/10.23919/FMCAD.2017.8102249
32. Havelund, K., Rosu, G.: Synthesizing monitors for safety properties. In: Katoen, J., Stevens, P. (eds.) TACAS 2002. LNCS, vol. 2280, pp. 342–356. Springer (2002). https://doi.org/10.1007/3-540-46002-0_24
33. Havelund, K., Rosu, G.: An overview of the runtime verification tool Java PathExplorer. Form. Meth. in Sys. Des. **24**(2), 189–215 (2004). https://doi.org/10.1023/B:FORM.0000017721.39909.4b
34. Isberner, M., Howar, F., Steffen, B.: The open-source learnlib - A framework for active automata learning. In: Kroening, D., Pasareanu, C.S. (eds.) CAV 2015 (Part I). LNCS, vol. 9206, pp. 487–495. Springer (2015). https://doi.org/10.1007/978-3-319-21690-4_32
35. Jaksic, S., Bartocci, E., Grosu, R., Nguyen, T., Nickovic, D.: Quantitative monitoring of STL with edit distance. Formal Methods in System Design **53**(1), 83–112 (2018). https://doi.org/10.1007/s10703-018-0319-x
36. Kretínský, J., Meggendorfer, T., Sickert, S.: Owl: A library for ω-words, automata, and LTL. In: Lahiri, S.K., Wang, C. (eds.) ATVA 2018. LNCS, vol. 11138, pp. 543–550. Springer (2018). https://doi.org/10.1007/978-3-030-01090-4_34
37. Kupferman, O., Vardi, M.Y.: Model checking of safety properties. Form. Meth. in Sys. Des. **19**(3), 291–314 (2001). https://doi.org/10.1023/A:1011254632723
38. Lee, I., Kannan, S., Kim, M., Sokolsky, O., Viswanathan, M.: Runtime assurance based on formal specifications. In: Arabnia, H.R. (ed.) PDPTA 1999. pp. 279–287. CSREA Press (1999)
39. Maler, O., Nickovic, D.: Monitoring temporal properties of continuous signals. In: Lakhnech, Y., Yovine, S. (eds.) FORMATS and FTRTFT 2004. LNCS, vol. 3253, pp. 152–166. Springer (2004). https://doi.org/10.1007/978-3-540-30206-3_12

40. Maler, O., Nickovic, D., Pnueli, A.: Checking temporal properties of discrete, timed and continuous behaviors. In: Avron, A., Dershowitz, N., Rabinovich, A. (eds.) Pillars of Computer Science, Essays Dedicated to Boris (Boaz) Trakhtenbrot on the Occasion of His 85th Birthday. LNCS, vol. 4800, pp. 475–505. Springer (2008). https://doi.org/10.1007/978-3-540-78127-1_26
41. Maler, O., Pnueli, A.: Timing analysis of asynchronous circuits using timed automata. In: Camurati, P., Eveking, H. (eds.) CHARME 1995. LNCS, vol. 987, pp. 189–205. Springer (1995). https://doi.org/10.1007/3-540-60385-9_12
42. Manna, Z., Pnueli, A.: Temporal verification of reactive systems - safety. Springer (1995)
43. Medhat, R., Bonakdarpour, B., Fischmeister, S., Joshi, Y.: Accelerated runtime verification of LTL specifications with counting semantics. In: Falcone and Sánchez [24], pp. 251–267. https://doi.org/10.1007/978-3-319-46982-9_16
44. Moosbrugger, P., Rozier, K.Y., Schumann, J.: R2U2: monitoring and diagnosis of security threats for unmanned aerial systems. Form. Meth. in Sys. Des. **51**(1), 31–61 (2017). https://doi.org/10.1007/s10703-017-0275-x
45. Ozay, N., Prabhakar, P. (eds.): HSCC 2019. ACM (2019)
46. Pike, L., Goodloe, A., Morisset, R., Niller, S.: Copilot: A hard real-time runtime monitor. In: Barringer, H., Falcone, Y., Finkbeiner, B., Havelund, K., Lee, I., Pace, G.J., Rosu, G., Sokolsky, O., Tillmann, N. (eds.) RV 2010. LNCS, vol. 6418, pp. 345–359. Springer (2010). https://doi.org/10.1007/978-3-642-16612-9_26
47. Pnueli, A., Zaks, A.: PSL model checking and run-time verification via testers. In: Misra, J., Nipkow, T., Sekerinski, E. (eds.) FM 2006. LNCS, vol. 4085, pp. 573–586. Springer (2006). https://doi.org/10.1007/11813040_38
48. Roesch, M.: Snort: Lightweight intrusion detection for networks. In: Parter, D.W. (ed.) LISA 1999. pp. 229–238. USENIX (1999)
49. Schwoon, S., Esparza, J.: A note on on-the-fly verification algorithms. In: Halbwachs, N., Zuck, L.D. (eds.) TACAS 2005. LNCS, vol. 3440, pp. 174–190. Springer (2005). https://doi.org/10.1007/978-3-540-31980-1_12
50. Tabuada, P., Neider, D.: Robust linear temporal logic. In: Talbot, J., Regnier, L. (eds.) CSL 2016. LIPIcs, vol. 62, pp. 10:1–10:21. Schloss Dagstuhl - LZI (2016). https://doi.org/10.4230/LIPIcs.CSL.2016.10
51. Zhang, X., Leucker, M., Dong, W.: Runtime verification with predictive semantics. In: Goodloe, A., Person, S. (eds.) NFM 2012. LNCS, vol. 7226, pp. 418–432. Springer (2012). https://doi.org/10.1007/978-3-642-28891-3_37

A Appendix

A.1 Proofs Omitted from Section 3

In this first part of the appendix, we present the proofs omitted in Section 3 on basic properties of the robust monitoring semantics.

An Alternative Definition of Robust Semantics for LTL We begin by introducing an alternative definition of the semantics of rLTL, which is more convenient to prove the results from Section 3. In fact, what follows is the original definition of the rLTL semantics and proceeds to define the evaluation function V_r by induction over the structure of formulas [50]. The definition we present in the main part has been introduced in later works of the original authors [6,5].

We need to begin by introducing some notation. For $\sigma = \sigma(0)\sigma(1)\sigma(2)\cdots \in \Sigma^\omega$ and a natural number n, define $\sigma[n,\infty) = \sigma(n)\sigma(n+1)\sigma(n+2)\cdots$, i.e., as the suffix of σ obtained by removing the first n letters of σ. Furthermore, we order the truth values in $\mathbb{B}_4$ by $0000 < 0001 < 0011 < 0111 < 1111$.

Following Tabuada and Neider's original work introducing rLTL its semantics can also be written as follows:

- $V_r(\sigma, p) = \begin{cases} 1111 & \text{if } p \in \sigma(0), \\ 0000 & \text{if } p \notin \sigma(0), \end{cases}$
- $V_r(\sigma, \neg\varphi) = \begin{cases} 1111 & \text{if } V_r(\sigma,\varphi) \neq 1111, \\ 0000 & \text{if } V_r(\sigma,\varphi) = 1111, \end{cases}$
- $V_r(\sigma, \varphi_1 \wedge \varphi_2) = \min\{V_r(\sigma,\varphi_1), V_r(\sigma,\varphi_2)\}$,
- $V_r(\sigma, \varphi_1 \vee \varphi_2) = \max\{V_r(\sigma,\varphi_1), V_r(\sigma,\varphi_2)\}$,
- $V_r(\sigma, \varphi_1 \rightarrow \varphi_2) = \begin{cases} 1111 & \text{if } V_r(\sigma,\varphi_1) \leq V_r(\sigma,\varphi_2), \\ V_r(\sigma,\varphi_2) & \text{if } V_r(\sigma,\varphi_1) > V_r(\sigma,\varphi_2), \end{cases}$
- $V_r(\sigma, \odot\varphi) = V_r(\sigma[1,\infty),\varphi)$,
- $V_r(\sigma, \diamondsuit\varphi) = \beta$ with $\beta[i] = \max_{n\geq 0} V_r(\sigma[n,\infty),\varphi)[i]$, $i \in \{1,\ldots,4\}$,
- $V_r(\sigma, \boxdot\varphi) = \beta$ with
  - $\beta[1] = \min_{n\geq 0} V_r(\sigma[n,\infty),\varphi)[1]$,
  - $\beta[2] = \max_{m\geq 0} \min_{n\geq m} V_r(\sigma[n,\infty),\varphi)[2]$,
  - $\beta[3] = \min_{m\geq 0} \max_{n\geq m} V_r(\sigma[n,\infty),\varphi)[3]$,
  - $\beta[4] = \max_{n\geq 0} V_r(\sigma[n,\infty),\varphi)[4]$,
- $V_r(\sigma, \varphi_1 \mathbf{U} \varphi_2) = \beta$ with

$$\beta[i] = \max_{n\geq 0} \min\{V_r(\sigma[n,\infty),\varphi_2)[i], \min_{0\leq n'<n} V_r(\sigma[n',\infty),\varphi_1)[i]\}, i \in \{1,\ldots,4\},$$

- $V_r(\sigma, \varphi_1 \mathbf{R} \varphi_2) = \beta$ with
  - $\beta[1] = \min_{n\geq 0} \max\{V_r(\sigma[n,\infty),\varphi_2)[1], \max_{0\leq n'<n} V_r(\sigma[n',\infty),\varphi_1)[1]\}$,
  - $\beta[2] = \max_{m\geq 0} \min_{n\geq m} \max\{V_r(\sigma[n,\infty),\varphi_2)[2], \max_{0\leq n'<n} V_r(\sigma[n',\infty),\varphi_1)[2]\}$,
  - $\beta[3] = \min_{m\geq 0} \max_{n\geq m} \max\{V_r(\sigma[n,\infty),\varphi_2)[3], \max_{0\leq n'<n} V_r(\sigma[n',\infty),\varphi_1)[3]\}$, and
  - $\beta[4] = \max_{n\geq 0} \max\{V_r(\sigma[n,\infty),\varphi_2)[4], \max_{0\leq n'<n} V_r(\sigma[n',\infty),\varphi_1)[4]\}$.

To reduce the number of cases we have to consider in our inductive proofs, we note that the robust eventually and the robust always operator are syntactic sugar. Formally, we say that two rLTL formulas φ_1, φ_2 are equivalent, if $V_r(\sigma,\varphi_1) = V_r(\sigma,\varphi_2)$ for every $\sigma \in \Sigma^\omega$. Now, let $\top = p \vee \neg p$ and $\bot = p \wedge \neg p$ for some atomic proposition p. Then, the robust eventually and the robust always are, as usual, expressible in terms of the robust until and the robust release, respectively.

Remark 1 (Tabuada and Neider [50]).

1. $\diamondsuit\varphi$ and $\top \mathbf{U} \varphi$ are equivalent.

2. $\boxdot \varphi$ and $\perp \mathbf{R}\, \varphi$ are equivalent.

Finally, recall that the definition of the semantics presented in the main part is induced by the function ltl translating a position $i \in \{1, 2, 3, 4\}$ and an rLTL formula into an LTL formula such that

$$V_r(\sigma, \varphi) = V(\sigma, \mathrm{ltl}(1, \varphi))V(\sigma, \mathrm{ltl}(2, \varphi))V(\sigma, \mathrm{ltl}(3, \varphi))V(\sigma, \mathrm{ltl}(4, \varphi)).$$

Thus, to determine the truth value of an rLTL formula φ on σ, it suffices to determine the truth value of the LTL formulas $\mathrm{ltl}(i, \varphi)$ on σ. For certain formulas, $\mathrm{ltl}(i, \varphi)$ is obtained from φ by a very simple rewriting (see Table 1).

Remark 2. Let φ be an rLTL formula that

- only uses negation, conjunction, disjunction, next, eventually, and always, and
- has no always in the scope of a negation.

Then,

- $\mathrm{ltl}(1, \varphi)$ is equivalent to the formula obtained from φ by replacing every $\odot$ by $\bigcirc$, every $\Diamond\!\!\!\cdot\;$ by $\Diamond$, and every $\boxdot$ by $\Box$,
- $\mathrm{ltl}(2, \varphi)$ is equivalent to the formula obtained from φ by replacing every $\odot$ by $\bigcirc$, every $\Diamond\!\!\!\cdot\;$ by $\Diamond$, and every $\boxdot$ by $\Diamond\Box$,
- $\mathrm{ltl}(3, \varphi)$ is equivalent to the formula obtained from φ by replacing every $\odot$ by $\bigcirc$, every $\Diamond\!\!\!\cdot\;$ by $\Diamond$, and every $\boxdot$ by $\Box\Diamond$, and
- $\mathrm{ltl}(4, \varphi)$ is equivalent to the formula obtained from φ by replacing every $\odot$ by $\bigcirc$, every $\Diamond\!\!\!\cdot\;$ by $\Diamond$, and every $\boxdot$ by $\Diamond$.

The Main Lemma We begin by proving a technical lemma about restrictions on the truth values that formulas can assume on simple words. These are later used to prove that certain truth values are not realizable in rLTL.

Lemma 5. *Let φ be an rLTL formula.*

1. $V_r(u\emptyset^\omega, \varphi)[2] = V_r(u\emptyset^\omega, \varphi)[3]$ *for all* $u \in \Sigma^*$.
2. $V_r(u^\omega, \varphi)[3] = V_r(u^\omega, \varphi)[4]$ *for all non-empty* $u \in \Sigma^*$.
3. *If φ does not contain the release operator, then* $V_r(u^\omega, \varphi)[1] = V_r(u^\omega, \varphi)[2]$ *for all non-empty* $u \in \Sigma^*$.

Proof. The proofs of all three items proceed by induction over the construction of φ. The induction start and the induction steps for Boolean connectives can be abstracted into the following closure property, which follows easily from the alternative definition of V_r:

> Let $T \subseteq \mathbb{B}_4$ contain 0000 and 1111. If $V_r(\sigma, \varphi_1)$ and $V_r(\sigma, \varphi_2)$ are in T, then so are $V_r(\sigma, p)$ for atomic propositions p, $V_r(\sigma, \neg\varphi_1)$, $V_r(\sigma, \varphi_1 \wedge \varphi_2)$, $V_r(\sigma, \varphi_1 \vee \varphi_2)$, and $V_r(\sigma, \varphi_1 \rightarrow \varphi_2)$.

1.) The induction start and the induction step for the Boolean operators follow from the closure property, where we pick T to be the set of truth values from $\mathbb{B}_4$ whose second and third bit coincide. Furthermore, due to Remark 1, we only have to consider the inductive steps for the next, until, and release operator. All three cases rely on the following simple fact: A suffix $u\emptyset^\omega[n, \infty)$ for some n is again of the form $u'\emptyset^\omega$, i.e., the induction hypothesis is applicable to suffixes. Also, if $n \geq |u|$, then $u\emptyset^\omega[n, \infty) = \emptyset^\omega$. In particular, $u\emptyset^\omega$ has only finitely many distinct suffixes.

So, first consider a formula of the form $\varphi = \odot\varphi_1$. Then, we have, for an arbitrary $u \in \Sigma^*$,

$$\begin{aligned} V_r(u\emptyset^\omega, \varphi)[2] =& V_r(u\emptyset^\omega[1, \infty), \varphi_1)[2] \\ =& V_r(u\emptyset^\omega[1, \infty), \varphi_1)[3] = V_r(u\emptyset^\omega, \varphi)[3], \end{aligned}$$

where the second equality is due to the induction hypothesis being applied to the suffix $u\emptyset^\omega[1,\infty)$.

Next, consider a formula of the form $\varphi = \varphi_1 \mathbf{U} \varphi_2$. Then, we have, for an arbitrary $u \in \Sigma^*$,

$$\begin{aligned}
&V_r(u\emptyset^\omega, \varphi)[2] \\
=&\max_{n\geq 0} \min\{V_r(u\emptyset^\omega[n,\infty), \varphi_2)[2], \min_{0\leq n'<n} V_r(u\emptyset^\omega[n',\infty), \varphi_1)[2]\} \\
=&\max_{n\geq 0} \min\{V_r(u\emptyset^\omega[n,\infty), \varphi_2)[3], \min_{0\leq n'<n} V_r(u\emptyset^\omega[n',\infty), \varphi_1)[3]\} \\
=&V_r(u\emptyset^\omega, \varphi)[3],
\end{aligned}$$

where the second equality follows from an application of the induction hypothesis to the suffixes $u\emptyset^\omega[n,\infty)$ and $u\emptyset^\omega[n',\infty)$.

It remains to consider a formula of the form $\varphi = \varphi_1 \mathbf{R} \varphi_2$. Then, we have, for an arbitrary $u \in \Sigma^*$, that $V_r(u\emptyset^\omega, \varphi)[2]$ is by definition equal to

$$\begin{aligned}
&\max_{m\geq 0} \min_{n\geq m} \max\{V_r(u\emptyset^\omega[n,\infty), \varphi_2)[2], \max_{0\leq n'<n} V_r(u\emptyset^\omega[n',\infty), \varphi_1)[2]\} \\
=&\max_{m\geq 0} \min_{n\geq m} \max\{V_r(u\emptyset^\omega[n,\infty), \varphi_2)[3], \max_{0\leq n'<n} V_r(u\emptyset^\omega[n',\infty), \varphi_1)[3]\} \\
=&\max_{m\geq |u|} \min_{n\geq m} \max\{V_r(u\emptyset^\omega[n,\infty), \varphi_2)[3], \max_{0\leq n'<n} V_r(u\emptyset^\omega[n',\infty), \varphi_1)[3]\} \\
=&\max_{m\geq |u|} \min_{n\geq m} \max\{V_r(\emptyset^\omega, \varphi_2)[3], \max_{0\leq n'\leq |u|} V_r(u\emptyset^\omega[n',\infty), \varphi_1)[3]\} \\
=&\max\{V_r(\emptyset^\omega, \varphi_2)[3], \max_{0\leq n'\leq |u|} V_r(u\emptyset^\omega[n',\infty), \varphi_1)[3]\},
\end{aligned}$$

The first equality follows from twice applying the induction hypothesis. For the second one, observe that

$$\min_{n\geq m} \max\{V_r(u\emptyset^\omega[n,\infty), \varphi_2)[3], \max_{0\leq n'<n} V_r(u\emptyset^\omega[n',\infty), \varphi_1)[3]\}$$

is increasing in m. For the third one, note that for all $n \geq |u|$, $u\emptyset^\omega[n,\infty) = \emptyset^\omega$, which means that we have eliminated every occurrence of m and n. This explains the last equality. Similarly, $V_r(u\emptyset^\omega, \varphi)[3]$ is by definition equal to

$$\begin{aligned}
&\min_{m\geq 0} \max_{n\geq m} \max\{V_r(u\emptyset^\omega[n,\infty), \varphi_2)[3], \max_{0\leq n'<n} V_r(u\emptyset^\omega[n',\infty), \varphi_1)[3]\} \\
=&\max\{V_r(\emptyset^\omega, \varphi_2)[3], \max_{0\leq n'\leq |u|} V_r(u\emptyset^\omega[n',\infty), \varphi_1)[3]\},
\end{aligned}$$

where the equality again follows from all suffixes $u\emptyset^\omega[n,\infty)$ with $n \geq |u|$ being equal to $\emptyset^\omega$. Thus, we have derived the desired equality between $V_r(u\emptyset^\omega, \varphi)[2]$ and $V_r(u\emptyset^\omega, \varphi)[3]$.

2.) The induction start and the induction steps for Boolean operators follow from the closure property, where we here pick T to be the set of truth values from $\mathbb{B}_4$ whose third and fourth bit coincide. For $u = u(0)\cdots u(|u|-1)$ and $n < |u|$, we define $\rho(u,n) = u(n)\cdots u(|u|-1)u(0)\cdots u(n-1)$, i.e., $\rho(u,n)$ is obtained by "rotating" u n times. The induction steps for the temporal operators are based on the following simple fact: The suffix $u^\omega[n,\infty)$ is equal to $(\rho(u, n \bmod |u|))^\omega$, i.e., the induction hypothesis is applicable to the suffixes. In particular, u^ω has only finitely many distinct suffixes, which all appear infinitely often in a cyclic order.

Now, the induction steps for the next and until operator are analogous to their counterparts in Item 1, as the only property we require there is that the induction hypothesis is applicable to suffixes. Hence, due to Remark 1, it only remains to consider the inductive step for the release operator.

So consider a formula of the form $\varphi = \varphi_1 \mathbf{R}\, \varphi_2$. Then, we have, for an arbitrary $u \in \Sigma^*$, that $V_r(u^\omega, \varphi)[3]$ is by definition equal to

$$\begin{aligned}
&\min_{m\geq 0} \max_{n\geq m} \max\{V_r(u^\omega[n,\infty), \varphi_2)[3], \max_{0\leq n'<n} V_r(u^\omega[n',\infty), \varphi_1)[3]\}\\
=&\min_{m\geq 0} \max_{n\geq m} \max\{V_r(u^\omega[n,\infty), \varphi_2)[4], \max_{0\leq n'<n} V_r(u^\omega[n',\infty), \varphi_1)[4]\}\\
=&\max_{0\leq n<|u|} \max\{V_r((\rho(u,n))^\omega, \varphi_2)[4], \max_{0\leq n'<n} V_r((\rho(u,n'))^\omega, \varphi_1)[4]\},
\end{aligned}$$

where the first equality follows from twice applying the induction hypothesis and the second one is due to all suffixes $u^\omega[n,\infty)$ being equal to $\rho(u, n \bmod |u|)^\omega$, and that there are only finitely many, which all appear infinitely often in a cyclic order among the $(\rho(u,n))^\omega$ for $0 \leq n < |u|$.

Similarly, $V_r(u^\omega, \varphi)[4]$ is by definition equal to

$$\begin{aligned}
&\max_{n\geq 0} \max\{V_r(u^\omega[n,\infty), \varphi_2)[4], \max_{0\leq n'<n} V_r(u^\omega[n',\infty), \varphi_1)[4]\}\\
=&\max_{0\leq n<|u|} \max\{V_r((\rho(u,n))^\omega, \varphi_2)[4], \max_{0\leq n'<n} V_r((\rho(u,n'))^\omega, \varphi_1)[4]\},
\end{aligned}$$

where the equality again follows from all suffixes $u^\omega[n,\infty)$ being equal to $\rho(u, n \bmod |u|)^\omega$, and that there are only finitely many, which appear in a cyclic order: In particular, after the first $|u|$ suffixes, we have seen all of them. Thus, we have derived the desired equality between $V_r(u^\omega, \varphi)[3]$ and $V_r(u^\omega, \varphi)[4]$.

3.) The induction start and the induction steps for Boolean operators are covered by the closure property, where we here pick T to be the set of truth values from $\mathbb{B}_4$ whose first and second bit coincide. The cases of the next and until operator are again analogous to the first and second item. Hence, we only have to consider the inductive step for the always operator, as we here only consider formulas without release.

So, consider a formula of the form $\varphi = \boxdot \varphi_1$. Here, we again rely on the fact that the suffix $u^\omega[n,\infty)$ is equal to $(\rho(u, n \bmod |u|))^\omega$. By definition, $V_r(u^\omega, \varphi)[1]$ is equal to

$$\begin{aligned}
\min_{n\geq 0} V_r(u^\omega[n,\infty), \varphi_1)[1] &= \min_{n\geq 0} V_r(u^\omega[n,\infty), \varphi_1)[2]\\
&= \min_{0\leq n<|u|} V_r((\rho(u,n))^\omega, \varphi_1)[2],
\end{aligned}$$

where the first equality is due to the induction hypothesis and the second one due to the fact that u^ω has only finitely many suffixes, which are all already realized by some $u^\omega[n,\infty)$ for $0 \leq n < |u|$.

Similarly, $V_r(u^\omega, \varphi)[2]$ is by definition equal to

$$\begin{aligned}
\max_{m\geq 0} \min_{n\geq m} V_r(u^\omega[n,\infty), \varphi_1)[1] &= \max_{m\geq 0} \min_{n\geq m} V_r(u^\omega[n,\infty), \varphi_1)[2]\\
&= \min_{0\leq n<|u|} V_r((\rho(u,n))^\omega, \varphi_1)[2],
\end{aligned}$$

where the two equalities follow as before: the first by induction hypothesis and the second one by the fact that u^ω has only finitely many suffixes, which all appear infinitely often in a cyclic order and which are all already realized by some $u^\omega[n,\infty)$ for $0 \leq n < |u|$. Thus, we have derived the desired equality between $V_r(u^\omega, \varphi)[1]$ and $V_r(u^\omega, \varphi)[2]$. ◻

Proof of Lemma 1 Recall that we need to prove that V_r^m is well-defined, i.e., that $V_r^m(u,\varphi) \in \mathbb{B}_4^?$ for every rLTL formula φ and every $u \in \Sigma^*$.

Proof. Let $V_r^m(u,\varphi)[i] = 0$ and $j < i$. By definition of V_r^m, we have $V_r(u\sigma, \varphi)[i] = 0$ for every $\sigma \in \Sigma^\omega$. Hence, due to the monotonicity of the truth values from $\mathbb{B}_4$ used to define V_r, we obtain $V_r(u\sigma, \varphi)[j] = 0$ for every such σ. Hence, $V_r^m(u,\varphi)[j] = 0$.

A dual argument shows that $V_r^m(u,\varphi)[i] = 1$ and $j > i$ implies $V_r^m(u,\varphi)[j] = 1$. Combining both properties yields $V_r^m(u,\varphi) \in 0^*?^*1^*$, i.e., $V_r^m(u,\varphi) \in \mathbb{B}_4^?$. ◻

Proof of Theorem 1 Recall that we need to prove that all truth values but 0011 and 0001 are realizable.

Proof. We begin by showing that 0011 and 0001 are not realizable.

First, towards a contradiction, assume there is an rLTL formula φ and a prefix u such that $V_r^m(u,\varphi) = 0011$, i.e., for every extension $u\sigma$, we have $V_r(u\sigma)[2] = 0$ and $V_r(u\sigma)[3] = 1$. However, by picking $\sigma = \emptyset^\omega$ we obtain the desired contradiction to Lemma 5.1.

The proof for 0001 is similar. Assume there is an rLTL formula φ and a prefix u such that $V_r^m(u,\varphi) = 0001$. Due to Lemma 2, we can assume that u is non-empty. Thus, we have $V_r(u^\omega,\varphi) = 0001$ by definition of V_r^m, which contradicts Lemma 5.2.

Finally, applying Lemma 5.3, one can show that no rLTL formula without the release operator realizes 0111. However, we show below that it is realizable by a formula with the release operator.

Next, we show that every other truth value $\beta \notin \{0011, 0001\}$ is indeed realizable. The witnessing pairs of prefixes and formulas are presented in Table 2.

First, consider $\beta = 0111$ with prefix $u = \emptyset\{a\}$ and formula $\varphi = a \,\mathbf{R}\, a$. We have $\mathrm{ltl}(1,\varphi) = a \,\mathbf{R}\, a$ and $\mathrm{ltl}(2,\varphi) = \Diamond\Box a \vee \Diamond a$. Note that $a \,\mathbf{R}\, a$ is violated by $u\sigma$, for every $\sigma \in \Sigma^\omega$. Dually, $\Diamond\Box a \vee \Diamond a$ is satisfied by $u\sigma$, for every $\sigma \in \Sigma^\omega$. Hence, for arbitrary $\sigma \in \Sigma^\omega$, we have $V_r(u\sigma,\varphi)[1] = 0$ and $V_r(u\sigma,\varphi)[2] = 1$. Hence, we have $V_r(u\sigma,\varphi) = 0111$ for every σ, as this is the only truth value that matches this pattern. Hence, by definition, we obtain $V_r^m(u,\varphi) = 0111$.

The verification for all other truth values is based on Remark 2, which is applicable to all formulas φ in the third column witnessing the realization of a truth value $\beta \neq 0111$. Now, for every such truth value β and corresponding pair (u,φ), one can easily verify the following:

- If $\beta[i] = 0$, then no $u\sigma$ satisfies $\mathrm{ltl}(i,\varphi)$.
- If $\beta[i] = 1$, then every $u\sigma$ satisfies $\mathrm{ltl}(i,\varphi)$.
- If $\beta[i] = ?$, then there are σ, σ' such that $u\sigma$ satisfies $\mathrm{ltl}(i,\varphi)$ and such that $u\sigma'$ violates $\mathrm{ltl}(i,\varphi)$. In all such cases, $\sigma,\sigma' \in \{\emptyset^\omega, \{a\}^\omega, \{a\}\emptyset^\omega, \emptyset\{a\}^\omega, (\{a\}\emptyset)^\omega\}$ suffice.

We leave the details of this slightly tedious, but trivial, verification to the reader. ◻

Proof of Lemma 2 Recall that we need to prove that $u \sqsubseteq u'$ implies $V_r^m(u,\varphi) \preceq V_r^m(u',\varphi)$.

Proof. Let $u \sqsubseteq u'$ and assume we have $V_r^m(u,\varphi)[i] \in \{0,1\}$. Thus, by definition, $V_r(u\sigma,\varphi)[i] = V_r^m(u,\varphi)[i]$ for every $\sigma \in \Sigma^\omega$. Now, as u is a prefix of u', we can decompose u' into $u' = uv$ for some $v \in \Sigma^*$ and every extension $u'\sigma'$ of u' is the extension $uv\sigma'$ of u. Hence, we have $V_r(u'\sigma',\varphi)[i] = V_r(uv\sigma',\varphi)[i] = V_r^m(u,\varphi)[i]$ for every $\sigma' \in \Sigma^\omega$. Thus, $V_r^m(u',\varphi)[i] = V_r^m(u,\varphi)[i]$.

As this property holds for every i, we obtain $V_r^m(u,\varphi) \preceq V_r^m(u',\varphi)$. ◻

Proof of Lemma 3 Here, we need to construct a formula φ and prefixes $u_0 \sqsubset u_1 \sqsubset u_2 \sqsubset u_3 \sqsubset u_4$ such that $V_r^m(u_0,\varphi) \prec V_r^m(u_1,\varphi) \prec V_r^m(u_2,\varphi) \prec V_r^m(u_3,\varphi) \prec V_r^m(u_4,\varphi)$.

Proof. Consider the sequence $\beta_0,\dots,\beta_4$ with $\beta_j = 0^j ?^{4-j}$ and note that we have $\beta_j \prec \beta_{j+1}$ for every $j < 4$. Furthermore, let $u_j = \emptyset^j$ for $j \in \{0,\dots,4\}$. We construct a formula φ such that $V_r^m(u_j,\varphi) = \beta_j$ for every $j \in \{0,\dots,4\}$.

To this end, let

- $\psi_{\beta_1} = \diamondsuit(a \wedge \boxdot\neg\diamondsuit a)$,
- $\psi_{\beta_2} = \boxdot(a \wedge \odot\neg a) \wedge \neg\diamondsuit\neg\diamondsuit a$, and
- $\psi_{\beta_3} = \diamondsuit\boxdot a \wedge \diamondsuit\neg\diamondsuit a$.

Later, we rely on the following fact about these formulas, which can easily be shown by applying Remark 2: we have $V_r^m(u,\psi_{\beta_j}) = \beta_j$ for every prefix u.

Further, for $j \in \{0,1,2,3\}$, let ψ_j be a formula that requires the proposition a to be violated at the first $j-1$ positions, but to hold at the j-th position (recall that we start counting at zero), i.e., $\psi_j = \bigwedge_{0\le j'<j} \odot^{j'} \neg a \wedge \odot^j a$.

Here, we define the nesting of next operators as usual: $\odot^0 \xi = \xi$ and $\odot^{j+1} \xi = \odot \odot^j \xi$. By definition, we have $V_r(\emptyset^{j+1}\sigma, \psi_j) = 0000$ for every $\sigma \in \Sigma^\omega$ (†).

Now, we define

$$\varphi = \psi_0 \vee \bigvee_{j=1}^{3} (\psi_{\beta_j} \wedge \psi_j)$$

and claim that it has the desired properties. To this end, we note that property (†) implies $V_r(\emptyset^4\sigma, \varphi) = 0000$ for every $\sigma \in \Sigma^\omega$ (††), as every disjunct of φ contains a conjunct of the form ψ_j for some $j \leq 3$. Also, let us mention that Remark 2 is applicable to φ.

It remains to prove $V_r^m(u_j, \varphi) = \beta_j$ for every $j \in \{0, \ldots, 4\}$.

- For $j = 0$, we have $u_0 = \varepsilon$ and $\beta_0 = ????$. Hence, it suffices to present $\sigma_0, \sigma_1 \in \Sigma^\omega$ such that $V_r(\sigma_0, \varphi) = 0000$ and $V_r(\sigma_1, \varphi) = 1111$.
  Due to property (††), we can pick $\sigma_0 = \emptyset^\omega$. To conclude, we pick $\sigma_1 = \{a\}^\omega$, as we have

  $$V_r(\sigma_1, \varphi) \geq V_r(\sigma_1, \psi_0) = V_r(\{a\}^\omega, a) = 1111,$$

  where the first inequality follows from ψ_0 being a disjunct of φ.
- For $j = 1$, we have $u_1 = \emptyset$ and $\beta_1 = 0???$. To show $V_r^m(u_1, \varphi) = \beta_1$, it suffices to present $\sigma_0, \sigma_1 \in \Sigma^\omega$ such that $V_r(u_1\sigma_0, \varphi) = 0000$, $V_r(u_1\sigma_1, \varphi) = 0111$, and show that $V_r(u_1\sigma, \varphi)[1] = 0$ for every $\sigma \in \Sigma^\omega$. For the former task, we again pick $\sigma_0 = \emptyset^\omega$ due to property (††). For the latter task, using Remark 2, one can easily verify that $\mathrm{ltl}(1, \varphi)$ is not satisfied by $u_1\sigma$ for any $\sigma \in \Sigma^\omega$, which implies the result.
  To conclude, consider

  $$\begin{aligned} & V_r(u_1\{a\}^\omega, \psi_{\beta_j} \wedge \psi_j) \\ =& \min\{V_r(u_1\{a\}^\omega, \psi_{\beta_j}), V_r(u_1\{a\}^\omega, \psi_j)\} \\ =& \min\{0111, 1111\} = 0111, \end{aligned}$$

  where $V_r(u_1\{a\}^\omega, \psi_{\beta_j}) = 0111$ can easily be verified using Remark 2.
- The reasoning for $j = 2, 3$ is along the same lines as the one for $j = 1$ and is left to the reader.
- For $j = 4$, we have $u_4 = \emptyset\emptyset\emptyset\emptyset$ and $\beta_4 = 0000$. Hence, our claim follows directly from property (††), which shows $V_r(u_4\sigma, \varphi) = 0000$ for every $\sigma \in \Sigma^\omega$. ☐

A.2 Proofs Omitted from Section 4

Proof of Lemma 4 Recall that we need to show that $u \in \mathcal{L}(\mathcal{B}_\beta^\varphi)$ if and only if there exists an infinite word $\sigma \in \Sigma^\omega$ with $V_r(u\sigma, \varphi) = \beta$.

Proof. We show both directions of Lemma 4 separately.

From left to right: Assume $u \in \mathcal{L}(\mathcal{B}_\beta^\varphi)$. Moreover, let $q \in F_\beta^\star$ be the final state reached by $\mathcal{B}_\beta^\varphi$ on an accepting run on u (which exists since $u \in \mathcal{L}(\mathcal{B}_\beta^\varphi)$). By definition of $F_\beta^\star$, this means that $\mathcal{L}(\mathcal{A}_\beta^\varphi(q)) \neq \emptyset$, say $\sigma \in \mathcal{L}(\mathcal{A}_\beta^\varphi(q))$. Since $\mathcal{A}_\beta^\varphi$ and $\mathcal{B}_\beta^\varphi$ share the same transition structures, the run of $\mathcal{B}_\beta^\varphi$ on u is also a run of $\mathcal{A}_\beta^\varphi$ on u, which both lead to state q. Therefore, $u\sigma \in \mathcal{L}(\mathcal{A}_\beta^\varphi)$. By Theorem 2, this is equivalent to $V_r(u\sigma, \varphi) = \beta$.

From right to left: Let $u \in \Sigma^*$ and $\sigma \in \Sigma^\omega$ such that $V(u\sigma, \varphi) = \beta$. By Theorem 2, we have $u\sigma \in \mathcal{L}(\mathcal{A}_\beta^\varphi)$. Consider an accepting run of $\mathcal{A}_\beta^\varphi$ on $u\sigma$, and let q be the state that $\mathcal{A}_\beta^\varphi$ reaches after reading the finite prefix u. Since $u\sigma \in \mathcal{L}(\mathcal{A}_\beta^\varphi)$, this means that $\sigma \in \mathcal{L}(\mathcal{A}_\beta^\varphi(q))$. Thus, $q \in F_\beta^\star$ because $\mathcal{L}(\mathcal{A}_\beta^\varphi(q)) \neq \emptyset$. Moreover, since the run of $\mathcal{A}_\beta^\varphi$ on u is also a run of $\mathcal{B}_\beta^\varphi$ on u, the NFA $\mathcal{B}_\beta^\varphi$ can also reach state q after reading u. Therefore, $u \in \mathcal{L}(\mathcal{B}_\beta^\varphi)$ since $q \in F_\beta^\star$. ☐

Proof of Theorem 3 Here, we need to show that $\mathcal{M}_\varphi$ is an rLTL monitor for φ, i.e., that $\lambda_{\mathcal{M}_\varphi}(u) = V_r^m(u, \varphi)$.

Proof. First, we observe that ξ indeed produces a valid value of $\mathbb{B}_4^?$ (i.e., a truth value of the form $0^*?^*1^*$). This follows immediately from the definition of ξ and the fact that the truth values of rLTL are in 0^*1^*.

Next, we observe that $\mathcal{M}_\varphi$ reaches state $(q_1, q_2, q_3, q_4, q_5)$ after reading a word $u \in \Sigma^*$ if and only if for each $\beta_j \in \mathbb{B}_4$ the DFA $\mathcal{C}^\varphi_{\beta_j}$ reaches state q_j after reading u. A simple induction over the length of inputs fed to $\mathcal{M}_\varphi$ proves this.

Now, let us fix a word $u \in \Sigma^*$ and assume that $(q_1, q_2, q_3, q_4, q_5)$ is the state reached by $\mathcal{M}_\varphi$ after reading u. This means that each individual DFA $\mathcal{C}^\varphi_{\beta_j} = (Q'_{\beta_j}, q'_{I,\beta_j}, \delta'_{\beta_j}, F'_{\beta_j})$ reaches state q_j after reading u. Let now

$$B = \big\{\beta_j \in \mathbb{B}_4 \mid q_j \in F'_{\beta_j}, j \in \{1, \ldots, 5\}\big\}$$

as in the definition of the output function λ of $\mathcal{M}_\varphi$. By applying Lemma 4, we then obtain

$$\beta_j \in B \Leftrightarrow q_j \in F'_{\beta_j} \Leftrightarrow u \in L(\mathcal{C}^\varphi_{\beta_j}) \Leftrightarrow u \in L(\mathcal{B}^\varphi_{\beta_j}) \Leftrightarrow \exists \sigma \in \Sigma^\omega \colon V_r(u\sigma, \varphi) = \beta_j.$$

To conclude the proof, it is left to show that $\xi(B) = V_r^m(u, \varphi)$. We show this for each bit individually using a case distinction over the elements of $\mathbb{B}^? = \{0, ?, 1\}$. So as to clutter this proof not too much, however, we here only discuss the case of ?, while noting that the remaining two cases can be proven analogously. Thus, let $i \in \{1, \ldots, 4\}$. Then,

$$\begin{aligned} \xi(B)[i] = ? &\Leftrightarrow \exists \beta, \beta' \in B \colon \beta[i] = 0 \text{ and } \beta'[i] = 1 \\ &\Leftrightarrow \exists \sigma_0, \sigma_1 \in \Sigma^\omega \colon V_r(u\sigma_0, \varphi)[i] = 0 \text{ and } V_r(u\sigma_1, \varphi)[i] = 1 \\ &\Leftrightarrow V_r^m(u, \varphi)[i] = ?. \end{aligned}$$

Since $\lambda\big((q_1, q_2, q_3, q_4, q_5)\big) = \xi(B)$, the Moore machine $\mathcal{M}_\varphi$ indeed outputs $V_r^m(u, \varphi)$ for every word $u \in \Sigma^*$, which proves Theorem 3. $\square$